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Sub-Riemannian vs. Euclidean dimension comparison and fractal geometry on Carnot groups

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Dedicated to the memory of Juha Heinonen (1960–2007)[4]

Abstract

We solve Gromov's dimension comparison problem for Hausdorff and box counting dimension on Carnot groups equipped with a Carnot–Carathéodory metric and an adapted Euclidean metric. The proofs use sharp covering theorems relating optimal mutual coverings of Euclidean and Carnot–Carathéodory balls, and elements of sub-Riemannian fractal geometry associated to horizontal self-similar iterated function systems on Carnot groups. Inspired by Falconer's work on almost sure dimensions of Euclidean self-affine fractals we show that Carnot–Carathéodory self-similar fractals are almost surely horizontal. As a consequence we obtain explicit dimension formulae for invariant sets of Euclidean iterated function systems of polynomial type. Jet space Carnot groups provide a rich source of examples.

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1. Introduction

Carnot groups are simply connected nilpotent Lie groups with graded Lie algebra equipped with a left invariant metric of sub-Riemannian type. They arise as ideal boundaries of noncompact rank one symmetric spaces, and serve as both examples of, and local models at regular points for, general sub-Riemannian (Carnot–Carathéodory) manifolds. The key role played by Carnot groups became evident in the 1970s in a series of influential papers and monographs (such as [55,53,25]) following the address by E.M. Stein at the 1970 International Congress of Mathematicians in Nice. More recently, Carnot groups have played a significant role in motivating the development of analysis in metric measure spaces, see particularly the work of Heinonen and Koskela [33,34], Cheeger [15] and Ambrosio and Kirchheim [1,2]. In this respect Carnot groups serve as models for non-Euclidean examples of spaces where the above cited results can be tested. On the other hand, it is well known that tools of Carnot–Carathéodory analysis are also motivated by applications in control theory [50,9,38]. Recently, the rototranslation group (a sub-Riemannian manifold locally modelled on the Heisenberg group) has emerged as a mathematical model for the neurogeometry of the first layer of the mammalian visual cortex [16].

This paper develops a theory of self-similar fractal geometry in general Carnot groups. It continues our program in this area [5,4,7], which is one component in a worldwide endeavor investigating sub-Riemannian geometric measure theory, including theories of rectifiability and perimeter [14,26,27,17,44,3,40]; fractals and tilings [58,59]; and geometric analysis of nonsmooth domains [28,42,52,51].

Gromov [30,31] has advocated a program to study the intrinsic metric geometry of Carnot–Carathéodory (CC) spaces. The present work originated in our consideration of the following problem, posed in [31, §3.1] (see also [30, Problem 0.6.C]):

Problem 1.1 *(Gromov).* Let M be a manifold equipped with a horizontal distribution $\mathcal{H} \subset TM$ and sub-Riemannian (Carnot–Carathéodory) metric g_0 with associated distance function d_0. For each $k = 0, 1, \ldots, \dim M$, determine

$$\beta_k := \inf\bigl\{\dim_{d_0}^H S\colon\ S \subset M,\ S \text{ compact},\ \dim_{top} S = k\bigr\}. \tag{1.1}$$

Here $\dim_d^H$ denotes Hausdorff dimension in a metric space (X, d) and $\dim_{top}$ stands for the topological dimension.

More generally, one may ask for a characterization of the set

$$\bigl\{(k, \beta)\colon\ \exists S \subset M,\ S \text{ Borel},\ \dim_{top} S = k,\ \dim_{d_0}^H S = \beta\bigr\}. \tag{1.2}$$

In the case when M is a Carnot group, our main results (described below) imply that for each k, the set of values of $\dim_{d_0}^H S$, where S varies over Borel subsets of M of topological dimension k, is an interval. We conjecture that $\{\beta\colon\ \exists S \subset M,\ S \text{ Borel},\ \dim_{top} S = k,\ \dim_{d_0}^H S = \beta\} = [\beta_k, \dim_{d_0}^H M]$ for each k. See Remark 8.1 for additional remarks, conjectures and discussion.

The computation of β_k is in general extremely challenging. It is clear that $\beta_k = k$ for sufficiently small k, in fact, for any k such that M contains an isometrically embedded Riemannian k-manifold. Thus $\beta_k = k$ for $k = 1, \ldots, n$ when $M = \mathbb{H}^n$ is the nth Heisenberg group. On the other hand,

$$\beta_{\dim M - 1} = \dim_{d_0}^H M - 1 > \dim M - 1 \tag{1.3}$$

for regular sub-Riemannian but non-Riemannian manifolds M, and especially in nonabelian Carnot groups; see [30, §2.1]. In particular, $\beta_1 = 1$ and $\beta_2 = 3$ in the first Heisenberg group $\mathbb{H}^1$.

It is of interest to pose Problem 1.1 for restricted classes of subsets of M, for instance, for smooth submanifolds. Gromov [30, §0.6] gives an explicit formula for the CC Hausdorff dimension of a general submanifold of a (regular) sub-Riemannian manifold. We illustrate with the first Heisenberg group $\mathbb{H}^1$. Denoting by β the CC Hausdorff dimension of a smooth k-dimensional submanifold in $\mathbb{H}^1$, we observe that only the following pairs (k, β) can occur:

$$\bigl\{(0,0), (1,1), (1,2), (2,3), (3,4)\bigr\}. \tag{1.4}$$

Examples which realize each case in (1.4) are (respectively): singletons, horizontal curves, non-horizontal curves, smooth surfaces, and the entire space $\mathbb{H}^1$. The absence of the pair $(2,2)$ in this list indicates that there is no smooth surface in the first Heisenberg group which has dimension two with respect to the sub-Riemannian metric. This feature of the geometry reflects the non-integrability of the horizontal distribution as indicated by the failure of the Frobenius theorem. We discuss Gromov's dimension comparison problem for smooth submanifolds in more detail in Remark 8.2.

A more ambitious goal is the following problem.

Problem 1.2. Let M be as in Problem 1.1 and let g be a Riemannian metric which extends g_0. Determine explicitly the set of triples (k,α,β) arising as the topological, Riemannian Hausdorff and sub-Riemannian Hausdorff dimensions of Borel subsets of M. More precisely, compute

$$\Delta(M)=\left\{(k,\alpha,\beta)\colon \exists S\subset M,\ S \text{ Borel},\ \dim_{top} S=k,\ \dim_d^H S=\alpha,\ \dim_{d_0}^H S=\beta\right\},$$

where d denotes the distance function determined by g.

The set in (1.2) is the projection of $\Delta(M)$ in the (k,β)-plane. Alternatively, we may consider the projection of $\Delta(M)$ in the (α,β)-plane. Since the Hausdorff measures are Borel regular, we can drop the restriction to Borel sets at this stage. Thus we are led to the following problem:

Problem 1.3. Let M be as in Problem 1.1. Determine explicitly the set of pairs (α,β) arising as the Riemannian/sub-Riemannian Hausdorff dimensions of subsets of M. More precisely, compute

$$\Delta'(M):=\left\{(\alpha,\beta)\in\mathbb{R}^2\colon \exists S\subset M,\ \dim_d^H S=\alpha,\ \dim_{d_0}^H S=\beta\right\}. \tag{1.5}$$

Problem 1.3 is a foundational question in sub-Riemannian geometric measure theory which asks for a quantitative description of the discrepancy between the sub-Riemannian metric g_0 and any taming Riemannian metric g. In this paper, we give a complete solution to Problem 1.3 in the case when M is a Carnot group. We shall see that this problem asks which Riemannian α-dimensional subsets of M are *most nearly horizontal* (β is smallest for fixed α) and which are *most non-horizontal* (β is largest for fixed α). The intuitive meaning of the phrase "horizontal set" is a set which is tangent to the horizontal distribution in M. We emphasize, however, that our framework is that of general geometric measure theory, and the examples which we will construct are typically not smooth submanifolds from either the Euclidean or the sub-Riemannian viewpoint.

Let $\mathbb{G}$ be a Carnot group equipped with a sub-Riemannian metric, of topological dimension N and homogeneous dimension Q. (See Section 2 for a review of definitions and terminology.) We will determine explicit functions $\beta_\pm=\beta_\pm^{\mathbb{G}}:[0,N]\to[0,Q]$ so that

$$\Delta'(\mathbb{G})=\left\{(\alpha,\beta)\in[0,N]\times[0,Q]\colon \beta_-(\alpha)\leqslant\beta\leqslant\beta_+(\alpha)\right\}. \tag{1.6}$$

See Theorems 2.4 and 2.6.

The results of this paper extend our prior work [5,7] on the Heisenberg group $\mathbb{H}^1$. We recall from [5] and [7] that the solution to Problem 1.3 when $M=\mathbb{H}^1$ is

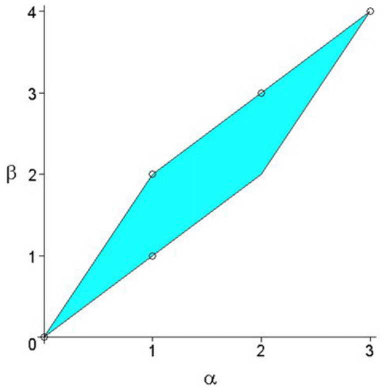


Fig. 1. Solution to Problem 1.3 in $\mathbb{H}^1$.

$$\Delta'(\mathbb{H}^1) = \{(\alpha, \beta) \in [0,3] \times [0,4]:\ \beta_-^{\mathbb{H}^1}(\alpha) = \max\{\alpha, 2\alpha - 2\} \leqslant \beta \leqslant \beta_+^{\mathbb{H}^1}(\alpha) = \min\{2\alpha, \alpha + 1\}\}. \tag{1.7}$$

See Fig. 1 for an illustration of (1.4) and (1.7). In Fig. 1 the set $\Delta'(\mathbb{H}^1)$ is represented by the shaded parallelogram while the points in (1.4) are represented by circled dots at the integer coordinates on the edges and corners. Notice the absence of $(2,2)$.

The solution to Problem 1.3 on a Carnot group $\mathbb{G}$ involves two stages. In the first stage we determine a region $\Delta'(\mathbb{G})$ in $\mathbb{R}^2$ where all possible dimension pairs are located. This stage utilizes precise mutual coverings of Euclidean, respectively Carnot–Carathéodory, balls which generalize the well-known Ball–Box Theorem [30,9]. In the second stage we prove a sharpness result: for any $(\alpha, \beta) \in \Delta'(\mathbb{G})$, there exists a compact set $S = S_{\alpha,\beta} \subset \mathbb{G}$ of topological dimension zero with $\dim_d^H S = \alpha$ and $\dim_{d_0}^H S = \beta$. To tackle the issue of sharpness we have to actually construct sets of prescribed Euclidean dimension whose Carnot–Carathéodory dimension is either as small or as large as possible as allowed by the first part of our result. Constructing sets of maximal Carnot–Carathéodory dimension is relatively straightforward while constructing sets with minimal Carnot–Carathéodory dimension is considerably harder. The difficulty is due to the non-integrability of the horizontal distribution.

The construction of examples demonstrating sharpness in our solution to Problem 1.3 relies on a theory of fractal geometry in Carnot groups. The development of such a theory is the second main goal of this paper. We shall consider self-similar iterated function systems and their invariant sets. The notion of self-similarity is understood here in terms of the Carnot–Carathéodory (CC) metric. The associated iterated function system will (typically) be a nonlinear, nonconformal system of polynomial type in the underlying Euclidean space. Let us mention that in our previous work [5] and [7] we also considered fractal sets in the setting of the Heisenberg group in connection with Gromov's problem. The iterated function systems we considered were affine Euclidean. Working in higher step Carnot groups, we have to deal with additional difficulties due to the non-linearity of the group law. One remarkable feature of our approach is that, as a byproduct of our investigations of sub-Riemannian self-similar fractals, we obtain exact formulas for the dimensions of invariant sets for a class of nonlinear, nonconformal Euclidean iterated function systems of polynomial type. These results are related to [20] and [22]. Example 2.10 (see also the discussion at the end of Subsection 4.2) and Section 7 indicate representative examples. Our approach provides a dramatic simplification over existing methods for computing such dimensions (see Falconer [23] for an approach using a nonconformal subadditive thermodynamic formalism). Our investigation of Carnot fractal geometry culminates in Theorem 2.8, which states, roughly speaking, that CC self-similar sets of prescribed sub-Riemannian dimension are almost surely horizontal sets (in the sense described above).

Our main results (Theorems 2.4, 2.6 and 2.8) hold also for box-counting dimension, see the discussion at the end of Section 3 and Remarks 4.5 and 4.17. Box-counting and Hausdorff dimension each play an important role in the study of attractors for general nonlinear iterated function systems, see [23]. It is a long-standing conjecture in dynamical systems that equality of these dimensions holds for such attractors in great generality, see [24] or [35]. The fact that our results hold for both Hausdorff and box-counting dimension, and that typically we obtain equality of these two values, is an essential feature of our approach with immediate applications to Euclidean fractal geometry.

The jet spaces $J^k(\mathbb{R}^m,\mathbb{R}^n)$ provide a rich source of examples of Carnot groups. In Section 6 we illustrate our results by discussing in detail the form which they take in the jet space context. We present a second Carnot group model for $J^k(\mathbb{R},\mathbb{R})$ in which left translation is an affine map in the underlying Euclidean geometry, whose linear part is given by a triangular matrix. In Subsection 6.3 we relate our work to recent work of Falconer and Miao [18] on almost sure dimensions of invariant sets of self-affine iterated function systems whose linear parts are given by upper triangular matrices. In Remark 6.5 we give the complete solution to Problem 1.2 in the second jet group $J^2(\mathbb{R},\mathbb{R})$.

The paper is structured as follows. In Section 2 we recall basic definitions, set notation and formulate our main results as Theorems 2.4, 2.6 and 2.8. Sections 3 and 4 contain the proofs of Theorems 2.4 and 2.6 respectively. In Section 5 we extend Falconer's almost sure dimension theory to the setting of Carnot self-similar fractals and prove Theorem 2.8. Section 6 discusses jet spaces, while Section 7 describes a more complicated example of a higher-step Carnot group. A concluding section (Section 8) presents additional remarks and open problems motivated by this work.

Some of the results of this paper were announced in [8].

2. Notation and statements of main results

2.1. *Carnot groups*

Let $(\mathbb{G},*)$ be a Carnot group with stratified Lie algebra $\mathfrak{g}=\mathfrak{v}_1\oplus\cdots\oplus\mathfrak{v}_s$ such that $[\mathfrak{v}_1,\mathfrak{v}_j]=\mathfrak{v}_{j+1}$, $j=1,\dots,s-1$, and $[\mathfrak{v}_1,\mathfrak{v}_s]=0$. The Euclidean space underlying $\mathbb{G}$ has dimension $N=\sum_{j=1}^s m_j$ while the homogeneous dimension of $\mathbb{G}$ is $Q=\sum_{j=1}^s jm_j$, where $\dim\mathfrak{v}_j=m_j$, and s is the step of the group. We denote by d_E the Euclidean metric in $\mathbb{G}$.

The map on $\mathfrak{g}$ which multiplies the elements of the jth stratum $\mathfrak{v}_j$ by j is a derivation. It generates a group of automorphic anisotropic dilations $\{\delta_r\colon r\in\mathbb{R}^+\}$ of $\mathfrak{g}$ defined by

$$\delta_r(U_1+\cdots+U_s)=rU_1+\cdots+r^sU_s,\quad U_j\in\mathfrak{v}_j,$$

with the property that $\delta_r\delta_t=\delta_{rt}$. We will also write δ_r for the corresponding automorphism $\exp\circ\delta_r\circ\log:\mathbb{G}\to\mathbb{G}$; here exp denotes the (bijective) exponential map and log denotes its inverse.

The exponential map $\exp : \mathfrak{g} \to \mathbb{G}$ relates also to the group operation in $\mathbb{G}$ via the Baker–Campbell–Hausdorff formula [54] as follows. For U and V in $\mathfrak{g}$

$$\exp(U) * \exp(V) = \exp\bigl(BCH(U, V)\bigr), \tag{2.1}$$

where

$$BCH(U, V) = U + V + \frac{1}{2}[U, V] + \frac{1}{12}\bigl(\bigl[U, [U, V]\bigr] - \bigl[V, [U, V]\bigr]\bigr) + \cdots.$$

Since exp is a bijection we may parametrize $\mathbb{G}$ by $\mathfrak{g}$. Exponential coordinates in $\mathbb{G}$ are defined as follows: denoting by $\{E_{jk}\colon j = 1, \ldots, s;\ k = 1, \ldots, m_j\}$ a graded orthonormal basis for $\mathfrak{g}$ (with respect to some fixed inner product) and by $\{e_{jk}\colon k = 1, \ldots, m_j\}$ the standard orthonormal basis of $\mathbb{R}^{m_j}$, we identify $x \in \mathbb{G}$ with the point $(x_1, \ldots, x_s) \in \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_s}$ where

$$x = \exp\left(\sum_{j=1}^{s} \sum_{k=1}^{m_j} \langle x_j, e_{jk}\rangle E_{jk}\right).$$

We denote by $\pi_j : \mathbb{G} \to \mathbb{R}^{m_j}$ the projection, given in exponential coordinates as $\pi_j(x_1, \ldots, x_s) = x_j$.

The Haar measure on $\mathbb{G}$, obtained by pushing forward the Lebesgue measure on $\mathfrak{g}$, is translation invariant. In exponential coordinates, this is just the Lebesgue measure on $\mathbb{R}^N$. If we denote by $|E|$ the measure of a set E, then $|\delta_r(E)| = r^Q|E|$.

We can identify the Lie algebra $\mathfrak{g}$ with the tangent space $T_o\mathbb{G}$ of $\mathbb{G}$ at the neutral element $o \in \mathbb{G}$. For $U \in \mathfrak{g}$ we have a unique left invariant vector field $X = X_U$ on $\mathbb{G}$ which agrees with U at o. Vector fields corresponding to vectors in $\mathfrak{v}_j$ span a vector bundle V_j over $\mathbb{G}$ of dimension m_j which varies smoothly from point to point. The hypothesis on the Lie algebra stratification implies that for all $j = 1, \ldots, s$ sections of V_j are obtained by taking linear combinations of commutators up to order j of vector fields in the first stratum V_1 (called the *horizontal distribution*). We denote by $H\mathbb{G}$ the horizontal distribution in $\mathbb{G}$.

Example 2.1. We model the Heisenberg group $\mathbb{H}^n$ with the polynomial group law on $\mathbb{R}^{2n+1}$ given by

$$p * q = \left(x_1 + y_1, \ldots, x_{2n} + y_{2n}, x_{2n+1} + y_{2n+1} + \frac{1}{2}\sum_{j=1}^{n}(x_j y_{n+j} - x_{n+j} y_j)\right),$$

where $p = (x_1, \ldots, x_{2n+1})$ and $q = (y_1, \ldots, y_{2n+1})$. This is a step two Carnot group of dimension $N = 2n + 1$ with stratified Lie algebra $\mathfrak{g} = \mathfrak{v}_1 \oplus \mathfrak{v}_2$, where $\mathfrak{v}_1$ and $\mathfrak{v}_2$ correspond to the vector bundles $V_1 = \operatorname{span}\{X_1, \ldots, X_{2n}\}$ and $V_2 = \operatorname{span}\{X_{2n+1}\}$,

$$X_j = \frac{\partial}{\partial x_j} - \frac{1}{2}x_{n+j}\frac{\partial}{\partial x_{2n+1}} \quad \text{and} \quad X_{n+j} = \frac{\partial}{\partial x_{n+j}} + \frac{1}{2}x_j\frac{\partial}{\partial x_{2n+1}} \quad \text{for } j = 1, \ldots, n,$$

and

$$X_{2n+1} = \frac{\partial}{\partial x_{2n+1}}.$$

The nontrivial commutation relations in $\mathfrak{g}$ are $[X_j, X_{n+j}] = X_{2n+1}$ for each $j = 1, \dots, n$. The homogeneous dimension of $\mathbb{H}^n$ is $Q = 2n + 2$.

Example 2.2. We model the Engel group $\mathbb{E}$ with the polynomial group law on $\mathbb{R}^4$ given by

$$x * y = \left(x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_2 y_1, x_4 + y_4 + x_3 y_1 + \frac{1}{2} x_2 y_1^2\right),$$

where $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$. This is a step three Carnot group of dimension $N = 4$ with stratified Lie algebra $\mathfrak{g} = \mathfrak{v}_1 \oplus \mathfrak{v}_2 \oplus \mathfrak{v}_3$, where $\mathfrak{v}_1$, $\mathfrak{v}_2$ and $\mathfrak{v}_3$ correspond to the vector bundles $V_1 = \operatorname{span}\{U_1, U_2\}$, $V_2 = \operatorname{span}\{V\}$, and $V_3 = \operatorname{span}\{W\}$,

$$U_1 = \frac{\partial}{\partial x_1}, \qquad U_2 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} + \frac{1}{2} x_1^2 \frac{\partial}{\partial x_4}, \qquad V = \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_4}, \quad \text{and} \quad W = \frac{\partial}{\partial x_4}.$$

The nontrivial commutation relations are $[U_1, U_2] = V$ and $[U_1, V] = W$. The homogeneous dimension of $\mathbb{E}$ can easily be calculated as $Q = 2 + 1 \cdot 2 + 1 \cdot 3 = 7$. The Engel group is isomorphic with the second jet group $J^2(\mathbb{R}, \mathbb{R})$; see Section 6 for a review of the Carnot structure of jet spaces.

2.2. *Carnot–Carathéodory metric*

We equip $\mathfrak{v}_1$ with an inner product $\langle \cdot, \cdot \rangle$ (for instance, by restricting the above inner product on $\mathfrak{g}$) and extend it as a left invariant inner product on V_1. The Carnot–Carathéodory (CC) metric d_{cc} is the standard sub-Riemannian metric defined using this inner product. For $x, y \in \mathbb{G}$, $d_{cc}(x, y)$ is the infimum of the lengths of all horizontal paths joining x and y. Here an absolutely continuous path $\gamma : [0, 1] \to \mathbb{G}$ is said to be *horizontal* if its tangents lie in the horizontal bundle V_1 almost everywhere, i.e., $\gamma'(t) \in (V_1)_{\gamma(t)}$ for almost every $t \in [0, 1]$, and the length of γ is $\int_0^1 \langle \gamma'(t), \gamma'(t) \rangle_{\gamma(t)}^{1/2} \, dt$. Note that because of the bracket generating property of V_1, and in view of Chow's theorem [9,30], every pair of points $x, y \in \mathbb{G}$ can be joined by a horizontal path, whence $d_{cc}(x, y)$ is finite.

The Carnot–Carathéodory metric is left invariant: $d_{cc}(x * y, x * z) = d_{cc}(y, z)$ for all $x, y, z \in \mathbb{G}$, and compatible with the dilations: $d_{cc}(\delta_r(x), \delta_r(y)) = r d_{cc}(x, y)$ for all $x, y \in \mathbb{G}$ and $r > 0$. We write $|x|_{cc} = d_{cc}(x, o)$ and $|x|_E = d_E(x, o)$. Observe that

$$|x|_E \leqslant |x|_{cc} \quad \text{for all } x \in \mathbb{G}, \tag{2.2}$$

with equality if $x = (x_1, 0, \dots, 0)$ in exponential coordinates (since in this case $\gamma : [0, 1] \to \mathbb{G}$, $\gamma(t) = \delta_t(x)$, is horizontal). An immediate consequence of (2.2) and (2.1) is the following fact:

$$\pi_1 : (\mathbb{G}, d_{cc}) \to \left(\mathbb{R}^{m_1}, d_E\right) \text{ is 1-Lipschitz.} \tag{2.3}$$

Note that π_j is never Lipschitz from $(\mathbb{G}, d_{cc})$ to $(\mathbb{R}^{m_j}, d_E)$ when $j \geqslant 2$, see [9] or [30].

The topology generated by the Carnot–Carathéodory metric is the same as that defined by the Euclidean metric on the underlying space. However the two metrics are never bi-Lipschitz equivalent if $s > 1$. If we denote by $B_{cc}(p, r)$ the CC ball centered at $p \in \mathbb{G}$ of radius $r > 0$ we see that $|B_{cc}(p, r)| = r^Q |B_{cc}(0, 1)|$ which implies that the Hausdorff dimension of $\mathbb{G}$ with respect to the CC metric is equal to Q. Evidently, $Q > N$ when $\mathbb{G}$ is nonabelian, i.e., $s > 1$. For example,

$$Q = 2n + 2 = \dim_{cc} \mathbb{H}^n > \dim_E \mathbb{H}^n = 2n + 1 = N.$$

In the case of the Engel group $\mathbb{E}$ the difference is even more dramatic:

$$Q = 7 = \dim_{cc} \mathbb{E} > \dim_E \mathbb{E} = 4 = N.$$

One of the main goals of this paper is to compare the Hausdorff dimensions of arbitrary subsets of arbitrary Carnot groups as measured with the Euclidean versus the CC metric.

2.3. Hausdorff and box-counting dimensions

In order to state our main results let us quickly recall for the sake of completeness the definitions of Hausdorff measure and Hausdorff and box-counting dimension in the general setting of a metric space (X, d). (For more information see [21,39,47].) Given $A \subset X$, the diameter of A is

$$\operatorname{diam}_{(X,d)}(A) = \sup\bigl\{d(x, y):\ x, y \in A\bigr\}.$$

We write $\operatorname{diam} = \operatorname{diam}_{(X,d)}$ when there is no risk of confusion, and abbreviate $\operatorname{diam}_E = \operatorname{diam}_{(\mathbb{G},d_E)}$ and $\operatorname{diam}_{cc} = \operatorname{diam}_{(\mathbb{G},d_{cc})}$.

For $0 \leqslant t < \infty$, $0 < \delta \leqslant \infty$ and $A \subset X$, the *t-dimensional Hausdorff premeasure* of A is

$$\mathcal{H}^t_{(X,d),\delta}(A) = \inf \sum_{i=1}^{\infty} \operatorname{diam}(A_i)^t,$$

where the infimum is taken over all coverings of A by sets $\{A_i\}$ with diameter at most δ. For fixed t and A, the quantity $\mathcal{H}^t_{(X,d),\delta}(A)$ is non-decreasing in δ; the quantity

$$\mathcal{H}^t_{(X,d)}(A) = \mathcal{H}^t_{(X,d),0}(A) := \sup_{\delta > 0} \mathcal{H}^t_{(X,d),\delta}(A)$$

is the *t-dimensional Hausdorff measure* of A. The *Hausdorff dimension* of A is

$$\dim^H_{(X,d)} A := \inf\bigl\{t \geqslant 0:\ \mathcal{H}^t_{(X,d)}(A) = 0\bigr\}.$$

As before we abbreviate $\mathcal{H}^t_{(\mathbb{G},d_E),\delta} = \mathcal{H}^t_{E,\delta}$ and $\mathcal{H}^t_{(\mathbb{G},d_{cc}),\delta} = \mathcal{H}^t_{cc,\delta}$ and write $\dim^H_E$, $\dim^H_{cc}$ for the corresponding Hausdorff dimensions.

Let us turn now to the definition of the box-counting dimension. For $\epsilon > 0$ and a bounded set $A \subset X$ we let $N_{(X,d)}(A, \epsilon)$ be the minimum number of sets of diameter ϵ needed to cover A. The *lower* (resp. *upper*) *box-counting dimension* of A is

$$\underline{\dim}^B_{(X,d)} A := \liminf_{\epsilon \to 0} \frac{\log N_{(X,d)}(A, \epsilon)}{\log 1/\epsilon} = \inf\bigl\{t:\ \underline{\mathcal{M}}^t(A) < \infty\bigr\}$$

where $\underline{\mathcal{M}}^t(A) = \liminf_{\delta\to 0} N_{(X,d)}(A,\delta)\delta^t$, resp.

$$\overline{\dim}^B_{(X,d)} A := \limsup_{\epsilon\to 0} \frac{\log N_{(X,d)}(A,\epsilon)}{\log 1/\epsilon} = \inf\{t\colon \overline{\mathcal{M}}^t(A) < \infty\}$$

where $\overline{\mathcal{M}}^t(A) = \limsup_{\delta\to 0} N_{(X,d)}(A,\delta)\delta^t$. Finally, the *box-counting dimension* of A is

$$\dim^B_{(X,d)} A = \lim_{\epsilon\to 0} \frac{\log N_{(X,d)}(A,\epsilon)}{\log 1/\epsilon}$$

if the limit exists. We abbreviate $N_{(\mathbb{G},d_E)}(A,\epsilon) = N_E(A,\epsilon)$, $N_{(\mathbb{G},d_{cc})}(A,\epsilon) = N_{cc}(A,\epsilon)$ and write $\underline{\dim}^B_E$, $\underline{\dim}^B_{cc}$, $\overline{\dim}^B_E$, $\overline{\dim}^B_{cc}$, $\dim^B_E$, $\dim^B_{cc}$ for the corresponding dimensions.

We record the basic estimates which relate Hausdorff and box counting dimensions in arbitrary metric spaces:

$$\dim^H_{(X,d)} A \leqslant \underline{\dim}^B_{(X,d)} A \leqslant \overline{\dim}^B_{(X,d)} A$$

for arbitrary bounded sets $A \subset X$. See, e.g., [21, (3.17)] for the case $X = \mathbb{R}^n$.

The bulk of this paper concerns Hausdorff dimension. To soften the notation we write $\dim_E = \dim^H_E$, $\dim_{cc} = \dim^H_{cc}$.

2.4. Statements of the main results and discussion

We define two functions $\beta_\pm$ which quantify the solution to Problem 1.3.

Definition 2.3. Let $\mathbb{G}$ be a step s Carnot group with stratified Lie algebra $\mathfrak{g} = \mathfrak{v}_1 \oplus \cdots \oplus \mathfrak{v}_s$. Denote by m_j the dimension of $\mathfrak{v}_j$, and by N (resp. Q) the topological (resp. homogeneous) dimension of $\mathbb{G}$. Let $m_0 = m_{s+1} = 0$. The *lower dimension comparison function for* $\mathbb{G}$ is the function $\beta_- = \beta_-^{\mathbb{G}} : [0,N] \to [0,Q]$ defined by

$$\beta_-(\alpha) = \sum_{j=0}^{\ell_-} j m_j + (1+\ell_-)\left(\alpha - \sum_{j=0}^{\ell_-} m_j\right), \tag{2.4}$$

where $\ell_- = \ell_-(\alpha) \in \{0,\ldots,s-1\}$ is the unique integer satisfying

$$\sum_{j=0}^{\ell_-} m_j < \alpha \leqslant \sum_{j=0}^{1+\ell_-} m_j. \tag{2.5}$$

The *upper dimension comparison function for* $\mathbb{G}$ is the function $\beta_+ = \beta_+^{\mathbb{G}} : [0,N] \to [0,Q]$ defined by

$$\beta_+(\alpha) = \sum_{j=\ell_+}^{s+1} j m_j + (-1+\ell_+)\left(\alpha - \sum_{j=\ell_+}^{s+1} m_j\right), \tag{2.6}$$

where $\ell_+ = \ell_+(\alpha) \in \{2, \ldots, s+1\}$ is the unique integer satisfying

$$\sum_{j=\ell_+}^{s+1} m_j < \alpha \leqslant \sum_{j=-1+\ell_+}^{s+1} m_j. \tag{2.7}$$

With this notation in place, our first result is the following.

Theorem 2.4. *In any Carnot group $\mathbb{G}$, we have*

$$\beta_-(\dim_E S) \leqslant \dim_{cc} S \leqslant \beta_+(\dim_E S) \tag{2.8}$$

for every $S \subset \mathbb{G}$. For bounded S, the inequalities in (2.8) *hold also if Hausdorff dimension is replaced by either upper or lower box-counting dimension.*

Let us comment on the formulae in (2.4) and (2.6). The first component $\sum_{j=0}^{\ell_-} j m_j$ in (2.4) can be interpreted as a weighted integer part of α with respect to the lowest possible strata in the stratification of the Lie algebra of $\mathbb{G}$. The second component $(1+\ell_-)(\alpha - \sum_{j=0}^{\ell_-} m_j)$ is the weighted fractional part of α with weight $1+\ell_-$. The upper dimension comparison function β_+ has a dual interpretation starting from the highest possible strata.

Remark 2.5. In the case when $M = \mathbb{G}$ is a Carnot group, the formula in [30, §0.6.B] for the CC Hausdorff dimension of a generic k-dimensional submanifold of a regular sub-Riemannian manifold M precisely coincides with $\beta_+^M(k)$.

The proof of Theorem 2.4 relies on certain optimal covering lemmas relating mutual coverings of Euclidean balls by Carnot–Carathéodory balls and vice versa. Such covering lemmas can be viewed as extensions and generalizations of the Ball–Box Theorem (Theorem 3.4).

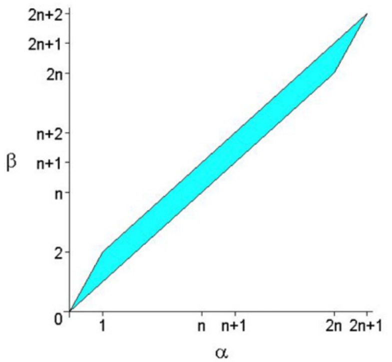
The sharpness of Theorem 2.4 is demonstrated in our next statement.

Theorem 2.6. *For all $0 \leqslant \alpha \leqslant N$ and $\beta_-(\alpha) \leqslant \beta \leqslant \beta_+(\alpha)$ there exists a bounded Borel set $S = S_{\alpha,\beta} \subset \mathbb{G}$ of topological dimension zero with $(\alpha, \beta) = (\dim_E S, \dim_{cc} S) = (\dim_E^B S, \dim_{cc}^B S)$. When $\beta = \beta_+(\alpha)$ we may choose $S_{\alpha,\beta}$ to be compact.*

Theorems 2.4 and 2.6 taken together yield (1.6). Note that the set $\Delta'(\mathbb{G})$ is always a convex polygon, since $\beta_\pm$ are monotone increasing and piecewise linear. Furthermore, $\beta_-(\alpha) \leqslant \beta_+(\alpha)$ and $\beta_+(\alpha) = Q - \beta_-(N-\alpha)$ for all $\alpha \in [0, N]$.

The solution to Problem 1.3 in Carnot groups depends only on the dimensions of the Lie algebra strata, and not on the algebraic relations which hold therein. By way of contrast, the solution to Problem 1.1 depends on these algebraic relations. We refer to Subsection 8.2 for further discussion of Problem 1.1.

Fig. 2 shows the solutions to Problem 1.3 in the Heisenberg and Engel groups: $\Delta'(\mathbb{H}^n)$ is the convex domain in $\mathbb{R}^2$ bounded by the graphs of the functions $\beta_+(\alpha) = \min\{2\alpha, \alpha+1\}$ and $\beta_-(\alpha) = \max\{\alpha, 2\alpha - 2n\}$, while $\Delta'(\mathbb{E})$ is the domain bounded by the graphs of the functions $\beta_+(\alpha) = \min\{3\alpha, 2\alpha+1, \alpha+3\}$ and $\beta_-(\alpha) = \max\{\alpha, 2\alpha-2, 3\alpha-5\}$. In Remark 6.5 we give the solution to Gromov's problem 1.1 in the Engel group.



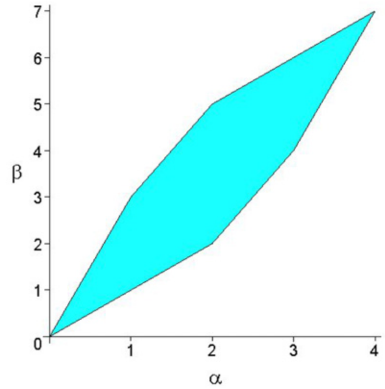


Fig. 2. Dimension comparison plot in (a) the Heisenberg group $\mathbb{H}^n$; (b) the Engel group $\mathbb{E}$.

To prove Theorem 2.6 we note that it suffices to construct the sets $S_{\alpha,\beta}$ in case $\beta = \beta_\pm(\alpha)$ and $\alpha \in [0, N]$. This follows from monotonicity of Hausdorff dimension and monotonicity of the functions $\beta_\pm$. Indeed, assume that such sets have been constructed in this case. Then, for an arbitrary $(\alpha, \beta) \in \Delta(\mathbb{G})$, the set

$$S_{\alpha,\beta} := S_{\alpha,\beta_-(\alpha)} \cup S_{(\beta_+)^{-1}(\beta),\beta}$$

satisfies $\dim_E S_{\alpha,\beta} = \dim_E^B S_{\alpha,\beta} = \alpha$ and $\dim_{cc} S_{\alpha,\beta} = \dim_{cc}^B S_{\alpha,\beta} = \beta$. The topological dimension of a union of two sets is the maximum of the individual topological dimensions provided one of the sets is closed [36, Theorem III.2]. Thus $\dim_{top} S_{\alpha,\beta} = 0$.

Intuitively a set S with $\dim_{cc} S = \beta_+(\dim_E S)$ tends to be as *vertical* as possible in that it lies in the direction of higher strata in the Lie algebra. In contrast, $\dim_{cc} S = \beta_-(\dim_E S)$ means that S is as *horizontal* as possible; S lies in the direction of lower strata. Vertical sets are relatively easy to find, while horizontal sets are considerably more challenging. The difficulty stems from the non-integrability of the horizontal distribution V_1. Horizontal sets in two step groups were first constructed by Strichartz [58,59] as L^∞ graphs. Our approach realizes such sets via fractal geometry. We consider invariant sets for iterated function systems (IFS) comprised of CC self-similarities. Such sets are naturally tangent to lower strata. In the construction of horizontal sets our starting point is the following proposition.

Proposition 2.7. *Let $F_1, \dots, F_M$ be contracting similarities of $\mathbb{G}$ in the form $F_i(p) = p_i * \delta_{r_i}(p)$ for some $p_i \in \mathbb{G}$ and $r_i < 1$. Let f_i be the projection of F_i in the first stratum, $f_i(p_1) = p_{i1} + r_i p_1$, and assume that the IFS $\{f_1, \dots, f_M\}$ on $\mathbb{R}^{m_1}$ satisfies the open set condition (see Subsection* 4.2 *for the definition). Let $\alpha \in (0, m_1]$ be the similarity dimension for the system $\{f_1, \dots, f_M\}$ and $\{F_1, \dots, F_M\}$, e.g., α is the unique solution to the equation $\sum_{i=1}^M r_i^\alpha = 1$. Then $0 < \mathcal{H}_E^\alpha(K)$ and $\mathcal{H}_{cc}^\alpha(K) < \infty$, where K denotes the invariant set for the IFS $\{F_1, \dots, F_M\}$. In particular, $\dim_E K = \dim_{cc} K = \alpha$.*

Proposition 2.7 generates horizontal sets in the lowest stratum ($0 \leqslant \alpha \leqslant m_1$). Note that in this range $\beta_-(\alpha) = \alpha$. To obtain horizontal sets in higher strata ($m_1 \leqslant \alpha \leqslant N$) as required by Theorem 2.6 we perform an iterative construction starting from a horizontal set S_{m_1} of dimension m_1

and taking successive extensions of the IFS to higher strata. The precise statements in this direction are Theorem 4.8 and Proposition 4.14 in Section 4.2 where we also review some basic results from the theory of iterated function systems that are needed for the proofs.

Proposition 2.7 also motivates our next result, on the almost sure horizontal nature of CC self-similar sets. While it is not true that arbitrary CC self-similar IFS in $\mathbb{G}$ satisfying the open set condition generate horizontal sets (as can be seen, for example, by considering Cantor sets along the vertical axis in $\mathbb{H}^1$), it is nevertheless true in a certain sense that generic IFS of this type have horizontal invariant sets. This claim is made more precise in the following theorem.

We consider CC self-similar IFS $\{F_1, \ldots, F_M\}$ on $\mathbb{G}$ consisting of maps of the form

$$F_i(p) = p_i * \delta_{r_i}(p), \quad i = 1, \ldots, M,$$

and denote by $\mathbf{r} = (r_1, \ldots, r_M) \in (0, 1)^M$ and $\mathbf{P} = (p_1, \ldots, p_M) \in \mathbb{G}^M$ the vectors of contraction ratios and translation parameters. We associate two numbers $\alpha = \alpha(\mathbf{r})$ and $\beta = \beta(\mathbf{r})$ as follows:

$$\beta(\mathbf{r}) = \min\{Q, t\}, \tag{2.9}$$

where t is the unique nonnegative value satisfying $\sum_{i=1}^{M} r_i^t = 1$, and

$$\alpha(\mathbf{r}) = (\beta_-)^{-1}\big(\beta(\mathbf{r})\big). \tag{2.10}$$

We write $K(\mathbf{P})$ for the invariant set of the IFS $\{F_1, \ldots, F_M\}$. Theorem 2.8 gives precise dimension formulas for $K(\mathbf{P})$ for almost every $\mathbf{P} \in \mathbb{G}^M$ with respect to the M-fold product Haar measure on $\mathbb{G}^M$.

Theorem 2.8. *Let $\mathbb{G}$ and $\mathbf{r}$ be as above, and let $\alpha = \alpha(\mathbf{r})$ and $\beta = \beta(\mathbf{r})$ be specified as in* (2.10) *and* (2.9). *If $r_i < \frac{1}{2}$ for all $i = 1, \ldots, M$, then the following statements hold:*

(a) $\dim_{cc} K(\mathbf{P}) \leqslant \beta$ *for all* $\mathbf{P} \in \mathbb{G}^M$,
(b) $\dim_E K(\mathbf{P}) \leqslant \alpha$ *for all* $\mathbf{P} \in \mathbb{G}^M$,
(c) $\dim_{cc} K(\mathbf{P}) = \beta$ *for a.e.* $\mathbf{P} \in \mathbb{G}^M$,
(d) $\dim_E K(\mathbf{P}) = \alpha$ *for a.e.* $\mathbf{P} \in \mathbb{G}^M$.

In particular, $\dim_{cc} K(\mathbf{P}) = \beta_-(\dim_E K(\mathbf{P}))$ *for a.e.* $\mathbf{P} \in \mathbb{G}^M$. *The same results hold if Hausdorff dimension is replaced by either upper or lower box-counting dimension, and the box-counting dimension exists for almost every* $\mathbf{P}$.

In informal terms, Theorem 2.8 asserts that generic self-similar sets of a fixed Euclidean Hausdorff dimension in a Carnot group, are horizontal sets. One can contrast this with Remark 2.5, according to which generic submanifolds of fixed dimension are maximally non-horizontal sets. Consider the collection of **all** subsets of a fixed Euclidean Hausdorff dimension in a Carnot group (or sub-Riemannian manifold). It would be interesting to understand the prevalence of horizontal or maximally non-horizontal sets within this collection.

Note the close relation between Theorems 2.4 and 2.8. Inequality (a) follows from the general theory of iterated function systems on metric spaces, and (b) follows directly from (a) and Theorem 2.4:

$$\dim_E K(\mathbf{P}) \leqslant (\beta_-)^{-1}\big(\dim_{cc} K(\mathbf{P})\big) \leqslant (\beta_-)^{-1}(\beta) = \alpha$$

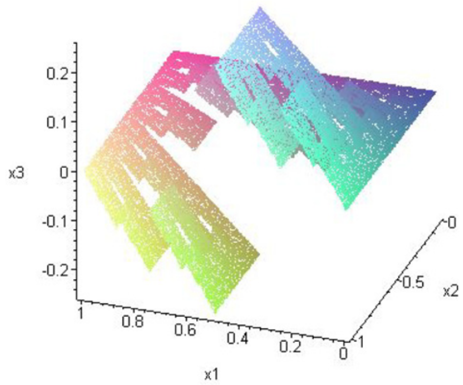


Fig. 3. The 2-adic Heisenberg square $Q_2 \subset \mathbb{H}^1$.

for every $\mathbf{P}$. Moreover, (c) follows directly from (d) and Theorem 2.4:

$$\dim_{cc} K(\mathbf{P}) \geqslant \beta_-\big(\dim_E K(\mathbf{P})\big) \geqslant \beta_-(\alpha) = \beta$$

for almost every $\mathbf{P}$. It thus suffices to prove (d), more precisely, to show that

$$\dim_E K(\mathbf{P}) \geqslant \alpha$$

for almost every $\mathbf{P} \in \mathbb{G}^M$. The (difficult) potential theoretic argument for this inequality is presented in Section 5.2; it utilizes ideas and techniques from the corresponding theory of almost sure dimensions of self-affine sets due to Falconer [20,22]. We note also that Theorem 2.8 provides another (albeit nonconstructive) approach to Theorem 4.8 and Proposition 4.14, especially for the construction of horizontal sets.

Example 2.9. We illustrate Proposition *2.7* with the b-adic *Heisenberg cube*. Fix a positive integer $b \geqslant 2$ and consider the following collection of b^2 contractive similarities:

$$F_{k_1k_2}: \mathbb{H}^1 \to \mathbb{H}^1, \qquad F_{k_1k_2}(p) = p_{k_1k_2} * \delta_{1/b}\big(p_{k_1k_2}^{-1} * p\big),$$

where $k_j \in \{0, \dots, b-1\}$ and $p_{k_1k_2} = (k_1, k_2, 0)$. Each such map is a similarity of $\mathbb{H}^1$ with contraction ratio b^{-1}, hence the collection $\{F_{k_1k_2}\}_{k_1,k_2=0,\dots,b-1}$ defines a unique nonempty compact *invariant set* $Q_b \subset \mathbb{H}^1$ characterized by the identity

$$Q_b = \bigcup_{k_1,k_2=0,\dots,b-1} F_{k_1k_2}(Q_b).$$

Then $\dim_E Q_b = \dim_{cc} Q_b = 2$. Fig. 3 shows the 2-adic Heisenberg square. Further analytical properties of the Heisenberg square and related fractals have been studied in detail in [4].

In a similar fashion we may consider the following collection of b^4 contractive similarities:

$$F_{k_1k_2k_3}: \mathbb{H}^1 \to \mathbb{H}^1, \qquad F_{k_1k_2k_3}(p) = p_{k_1k_2k_3} * \delta_{1/b}\big(p_{k_1k_2k_3}^{-1} * p\big),$$

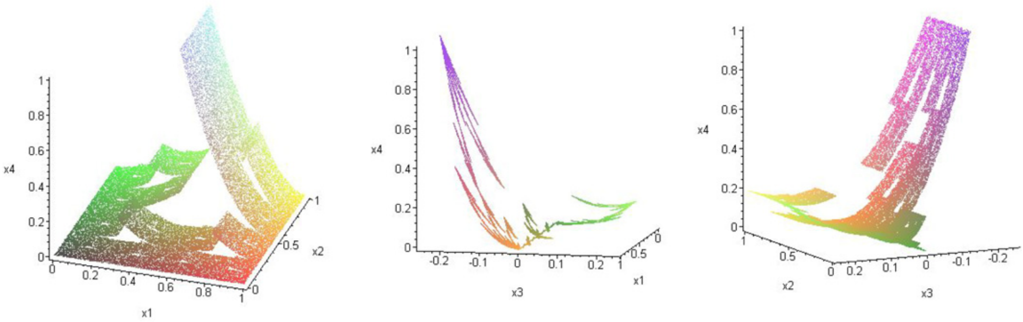


Fig. 4. 3-dimensional projections of the Engel square.

where $k_1, k_2 \in \{0, \dots, b-1\}$, $k_3 \in \{0, \dots, b^2-1\}$ and $p_{k_1k_2k_3} = (k_1, k_2, k_3)$. Again, each such map is a similarity of $\mathbb{H}^1$ with contraction ratio b^{-1} and the collection of these maps generates an invariant set $T_b \subset \mathbb{H}^1$ characterized by the identity

$$T_b = \bigcup_{\substack{k_1,k_2=0,\dots,b-1\\ k_3=0,\dots,b^2-1}} F_{k_1k_2k_3}(T_b). \tag{2.11}$$

Then

$$\dim_E T_b = 3 \tag{2.12}$$

and

$$\dim_{cc} T_b = 4. \tag{2.13}$$

Eq. (2.11) shows $\mathbb{H}^1$ may be tiled with congruent copies of T_b (we emphasize that congruence here refers to isometric copies in the **sub-Riemannian** metric). Note that this tiling is a self-affine fractal tiling in the underlying Euclidean geometry. Strichartz [58,59] was the first to consider tilings of this type in general two-step nilpotent Lie groups. See also Gelbrich [29].

Example 2.10. For further illustration, let us consider the following IFS generating an invariant set in $\mathbb{E}$ which we call the *Engel square*. With $x = (x_1, x_2, x_3, x_4)$ denoting a general element of $\mathbb{E}$ we note first that the Engel dilations take the form $\delta_r(x) = (rx_1, rx_2, r^2x_3, r^3x_4)$, while the group inverse of x is $(-x_1, -x_2, -x_3 + x_1x_2, -x_4 + x_1x_3 - \frac{1}{2}x_1^2x_2)$. Consider the IFS $F_1(x) = \delta_{1/2}(x)$, $F_2(x) = p_1 * \delta_{1/2}(p_1^{-1} * x)$, $F_3(x) = p_2 * \delta_{1/2}(p_2^{-1} * x)$, and $F_4(x) = p_1 * p_2 * \delta_{1/2}(p_2^{-1} * p_1^{-1} * x)$, where $p_1 = (1, 0, 0, 0)$ and $p_2 = (0, 1, 0, 0)$. It is clear that projection to the lowest stratum $\mathbb{R}^2$ gives a Euclidean IFS satisfying the open set condition whose invariant set is the unit square $[0, 1]^2$. Let us denote by Q the invariant set of $\{F_1, F_2, F_3, F_4\}$ which we call the Engel square. Then Proposition 2.7 gives $\dim_{cc} Q = \dim_E Q = 2$. Note that F_3 and F_4 are quadratic maps, see (4.37) and (4.38). In Fig. 4, we show the projections of Q in the hyperplanes $x_3 = 0$, $x_2 = 0$ and $x_1 = 0$. The projection of Q in the hyperplane $x_4 = 0$ coincides with the 2-adic Heisenberg square; see Section 6 for further details on the relation between the Heisenberg and Engel groups.

As demonstrated in Example 2.10, an interesting corollary of Proposition 2.7, its more general cousin Proposition 4.14, and Theorem 2.8 is a formula for the dimensions of invariant sets in the underlying Euclidean space for a certain class of nonlinear IFS which are not necessarily even generated by Euclidean contractions. According to the Baker–Campbell–Hausdorff formula, self-similarities of a step s Carnot group are polynomial maps of degree $s-1$. This provides a novel approach for dimension computation for a class of polynomial Euclidean IFS. In the Heisenberg group the relevant IFS are generated by affine maps. Dimension formulae for Euclidean self-affine sets have been obtained by Falconer [20,22], and for Heisenberg horizontal self-affine sets by the first two authors [7].

3. Proof of the dimension comparison theorem

Denote by $\mathcal{H}_E^\alpha$, resp. $\mathcal{H}_{cc}^\beta$ the α-, resp. β-dimensional Hausdorff measures with respect to the Euclidean, resp. CC, metric. The Hausdorff dimension statements in Theorem 2.4 are a consequence of the following inequalities relating these measures.

Proposition 3.1 *(Hausdorff measure comparison). Let $0 \leqslant \alpha \leqslant N$ and $\beta_\pm(\alpha)$ be as in Definition 2.3 and let $b>0$. There exists $L = L(\mathbb{G}, b)$ so that*

$$\mathcal{H}_{cc}^{\beta_+(\alpha)}(S)/L \leqslant \mathcal{H}_E^\alpha(S) \leqslant L\mathcal{H}_{cc}^{\beta_-(\alpha)}(S) \tag{3.1}$$

for all $S \subset B_{cc}(0,b)$, where $B_{cc}(0,R)$ denotes the CC ball of radius R centered at the identity $0 \in \mathbb{G}$.

The inequalities in (3.1) immediately imply those in (2.8). Proposition 3.1 is established with the aid of the following ball covering lemma (compare also the Exercise in Section 0.6.C of [30]):

Lemma 3.2 *(Covering Lemma). Let $K \subset \mathbb{G}$ be a bounded set.*

(a) *For each $\ell \in \{2,\dots,s\}$ there exists a constant $M_+ = M_+(\ell,K)$ such that every Euclidean ball with radius $0<r<1$ contained in K can be covered by a collection of CC balls with radius $r^{1/(\ell-1)}$ of cardinality no more than $M_+/r^{\lambda_+(\ell)}$, where*

$$\lambda_+(\ell) := \frac{1}{\ell-1}\sum_{j=\ell}^{s+1} jm_j - \sum_{j=\ell}^{s+1} m_j.$$

(b) *For each $\ell \in \{1,\dots,s-1\}$ there exists a constant $M_- = M_-(\ell,K)$ such that every CC ball with radius $0<r<1$ contained in K can be covered by a collection of Euclidean balls with radius $r^{\ell+1}$ of cardinality no more than $M_-/r^{\lambda_-(\ell)}$, where*

$$\lambda_-(\ell) := (\ell+1)\sum_{j=0}^{\ell} m_j - \sum_{j=0}^{\ell} jm_j.$$

For proving Lemma 3.2 we require some preliminary results. First we establish a Euclidean distortion estimate for left translation in Carnot groups.

Lemma 3.3 *(Euclidean distortion). Let K_1 and K_2 be bounded subsets of $\mathbb{G}$. Then there exists a constant $C_1(K_1, K_2)$ so that*

$$d_E(p * q, p * q_0) \leqslant C_1(K_1, K_2) d_E(q, q_0) \tag{3.2}$$

whenever $p \in K_1$ and $q, q_0 \in K_2$. In particular, if p and q are points in a bounded set $K \subset \mathbb{G}$, then

$$d_E\big(p^{-1} * q, 0\big) \leqslant C_1(K) d_E(q, p) \tag{3.3}$$

where $C_1(K) = C_1(K^{-1}, K)$, and

$$p^{-1} * B_E(p, r) \subseteq B_E\big(0, C_1(K) r\big). \tag{3.4}$$

Proof. Inequality (3.2) follows from the structure of the Baker–Campbell–Hausdorff formula, which implies that for fixed $p \in \mathbb{G}$, the coordinate expressions of the map $h : \mathbb{G} \to \mathbb{G}$ given by $h(q) = p * q - p * q_0$, are polynomials of degree at most $s - 1$ and $h(q_0) = 0$. Inequality (3.3) and inclusion (3.4) are easy consequences. □

In the proof of Lemma 3.2, we shall primarily work with boxes instead of balls. We recall below the notion of boxes in the Euclidean and Carnot metrics and their relation to balls.

The Euclidean box with center 0 and radius r is the N-cube $\mathrm{Box}_E(0, r) = [-r, r]^N$ and the Euclidean box with center $p \in G$ and radius r is the translated cube $\mathrm{Box}_E(p, r) = p + \mathrm{Box}_E(0, r)$. We introduce the Carnot box with center 0 and radius r as the set

$$\mathrm{Box}_{cc}(0, r) = [-r, r]^{m_1} \times \big[-r^2, r^2\big]^{m_2} \times \cdots \times \big[-r^s, r^s\big]^{m_s},$$

and the Carnot box with center $p \in \mathbb{G}$ and radius r as the translated box $\mathrm{Box}_{cc}(p, r) = p * \mathrm{Box}_{cc}(0, r)$.

Note that, for $r \ll 1$, the Carnot box is much flatter in non-horizontal directions than its Euclidean counterpart. In fact

$$\mathrm{Vol}\big(\mathrm{Box}_{cc}(0, r)\big) = 2^N r^Q \ll 2^N r^N = \mathrm{Vol}\big(\mathrm{Box}_E(0, r)\big).$$

Note also that the Carnot box with center $p \neq 0$ is twisted and not a Cartesian product as is the case for its Euclidean counterpart.

The fundamental result relating Carnot balls and Carnot boxes is the Ball–Box Theorem, see Montgomery [50, Theorem 2.10] or Gromov [30, 0.5.A]. For future reference, we also record the Ball–Box Theorem in the Euclidean setting.

Theorem 3.4 *(Ball–Box Theorem). For all $r > 0$, we have*

$$\mathrm{Box}_E(p, r/\sqrt{N}) \subset B_E(p, r) \subset \mathrm{Box}_E(p, r). \tag{3.5}$$

Moreover, there exists a constant $C_{BB} \geqslant 1$ so that

$$\mathrm{Box}_{cc}(p, r/C_{BB}) \subset B_{cc}(p, r) \subset \mathrm{Box}_{cc}(p, C_{BB} r) \tag{3.6}$$

for all $r > 0$.

The following covering theorem, see [47], [32, Chapter 1] is a useful tool for constructing efficient coverings with balls in metric spaces.

Theorem 3.5 *(5r Covering Theorem for Balls). Every family $\mathcal{F}$ of closed balls with uniformly bounded radius in a separable metric space X contains a pairwise disjoint subfamily $\mathcal{G}$ such that $\bigcup_{B\in\mathcal{F}} B \subset \bigcup_{B\in\mathcal{G}} 5B$, where $5B = B(p,5r)$ when $B = B(p,r)$ is the ball centered at $p \in X$ with radius $r > 0$.*

The proof of Lemma 3.2 uses the following covering theorem for Carnot boxes which is a straightforward consequence of the $5r$ Covering Theorem for balls and the Ball–Box Theorem.

Lemma 3.6 *(Covering Theorem for Boxes). Fix $r > 0$, then every subset $S \subset \mathbb{G}$ can be covered by a family of boxes $\{\mathrm{Box}_{cc}(p,r)\colon\ p \in S'\}$, where $S' \subset S$, so that the family $\{\mathrm{Box}_{cc}(p, r/5C_{BB}^2)\colon\ p \in S'\}$ is pairwise disjoint.*

Proof. Let $\mathcal{F} = \{B_{cc}(p, r/C_{BB})\colon\ p \in S\}$ and let $\mathcal{G} = \{B_{cc}(p, r/5C_{BB})\colon\ p \in S'\}$ be the pairwise disjoint subfamily whose existence is guaranteed by Theorem 3.5 applied in the metric space $(\mathbb{G}, d_{cc})$. Then it follows that

$$S \subset \bigcup_{p\in S'} B_{cc}(p, r/C_{BB}).$$

The Ball–Box Theorem, specifically (3.6), yields that $\{\mathrm{Box}_{cc}(p,r)\colon\ p \in S'\}$ is a covering of S, and also that $\{\mathrm{Box}_{cc}(p, r/5C_{BB}^2)\colon\ p \in S'\}$ is pairwise disjoint. This completes the proof. □

With these preparations at hand, we commence the proof of Lemma 3.2.

Proof of Lemma 3.2. We first prove (b) as (a) requires a more subtle argument due to the twisting involved in the definition of Carnot boxes. The proof of (b) is accomplished in two stages. Let $B_{cc}(p,r)$ be a CC ball with radius $0 < r < 1$. In the first stage we assume that $p = 0$ and estimate the number of Euclidean boxes $\mathrm{Box}_E(q, r^{\ell+1})$ needed to cover the Carnot box $\mathrm{Box}_{cc}(0,r)$ where the centers q lie in $\mathrm{Box}_{cc}(0,r)$. To do so, first observe that since $\mathrm{Box}_{cc}(0,r)$ is compact, we may assume the centers of these boxes lie in a finite set $I \subset \mathrm{Box}_{cc}(0,r)$. Next observe that both $\mathrm{Box}_{cc}(0,r)$ and $\mathrm{Box}_E(q, r^{\ell+1})$ have the structure of a Cartesian product of intervals. The sides of $\mathrm{Box}_E(q, r^{\ell+1})$ all have length $2r^{\ell+1}$, while the lengths of the sides of $\mathrm{Box}_{cc}(0,r)$ vary according to the strata dimensions of the Lie algebra of $\mathbb{G}$. To estimate the cardinality $\#I$ of I, we simply multiply together the number of intervals of length $2r^{\ell+1}$ needed to cover intervals of length $2r$ (m_1 times), $2r^2$ (m_2 times), and so on. Note that since $r < 1$, it follows that $2r^j \leqslant 2r^{\ell+1}$ when $j \geqslant \ell+1$, and so we only require one interval of length $2r^{\ell+1}$ to cover each of the intervals coming from the $(j+1)$st through sth strata of $\mathrm{Box}_{cc}(0,r)$. Thus

$$\begin{aligned}\#I &= \prod_{j=0}^{s}\left(\left[\frac{r^j}{r^{\ell+1}}\right]+1\right)^{m_j} = \prod_{j=0}^{\ell+1}\left(\left[\frac{r^j}{r^{\ell+1}}\right]+1\right)^{m_j} \\ &\leqslant \prod_{j=0}^{\ell+1}\left(\frac{r^j + r^{\ell+1}}{r^{\ell+1}}\right)^{m_j} \leqslant \prod_{j=0}^{\ell+1}\left(\frac{2r^j}{r^{\ell+1}}\right)^{m_j} = \frac{2^{\sum_{j=0}^{\ell+1} m_j}}{r^{\lambda_-(\ell)}}. \end{aligned} \tag{3.7}$$

An application of the Ball–Box Theorem completes the proof in the first stage.

In the second stage, we extend the above to general Carnot boxes $\mathrm{Box}_{cc}(p,r)$ contained in K. First, we note that (3.6), the statement from the first stage, (3.5), and Lemma 3.3 show that

$$\begin{aligned} B_{cc}\big(q, r/N^{\frac{1}{2(\ell+1)}} C_1(K)^{\frac{1}{\ell+1}} C_{BB}\big) &\subset \mathrm{Box}_{cc}\big(q, r/N^{\frac{1}{2(\ell+1)}} C_1(K)^{\frac{1}{\ell+1}}\big) \\ &\subset q * \Big(\bigcup_{p'\in I} \mathrm{Box}_E\big(p', r^{\ell+1}/N^{\frac{1}{2}} C_1(K)\big)\Big) \\ &\subset q * \Big(\bigcup_{p'\in I} B_E\big(p', r^{\ell+1}/C_1(K)\big)\Big) \subset \bigcup_{p'\in I} B_E\big(q*p', r^{\ell+1}\big) \end{aligned}$$

whenever $q \in K$. Since $B_{cc}(p,r)$ is compact, there is a finite subset $J \subset B_{cc}(p,r)$ such that

$$B_{cc}(p,r) \subset \bigcup_{q\in J} B_{cc}\big(q, r/2^{\frac{1}{2(\ell+1)}} C_1(K)^{\frac{1}{\ell+1}} C_{BB}\big) \subset \bigcup_{q\in J}\bigcup_{p'\in I} B_E\big(q*p', r^{\ell+1}\big).$$

By the above $5r$ covering theorem in combination with a volume counting argument we see that $\#J$ depends only on the constant $2^{\frac{1}{2(\ell+1)}} C_1(K)^{\frac{1}{\ell+1}} C_{BB}$. Letting $M_- = (\#J)2^{\sum_{j=0}^{\ell+1} m_j}$ and using (3.7) completes the proof of (b).

We now turn to the proof of (a). Let $B_E(p,r)$ be a Euclidean ball with radius $0<r<1$. In the first stage of the proof we assume again that $p=0$ and estimate the number of Carnot boxes of the form $\mathrm{Box}_{cc}(q, r^{\frac{1}{\ell-1}})$ that are required to cover the Euclidean box $\mathrm{Box}_E(0,r) = [-r,r]^N$. Since the Carnot boxes $\mathrm{Box}_{cc}(q, r^{\frac{1}{\ell-1}})$ are twisted and do not have a simple Cartesian structure, we cannot employ the rectilinear covering argument used in the proof of (b). Instead we use volume estimates arising from Lemma 3.6 in the following manner. First note that if $0<r<1$ and $\ell \in \{2,\dots,s\}$, then

$$\mathrm{Box}_{cc}\big(0, r^{\frac{1}{\ell-1}}\big) \supseteq [-r,r]^{\sum_{j=0}^{\ell-1} m_j} \times \big[-r^{\frac{\ell}{\ell-1}}, r^{\frac{\ell}{\ell-1}}\big]^{m_\ell} \times \cdots \times \big[-r^{\frac{s}{\ell-1}}, r^{\frac{s}{\ell-1}}\big]^{m_s}.$$

It follows that if we are to cover $\mathrm{Box}_E(0,r)$ with Carnot boxes of the form of $\mathrm{Box}_{cc}(q, r^{\frac{1}{\ell-1}})$, we need only consider centers q whose coordinates vanish up to the $(\ell-1)$st stratum, in particular

$$q \in \widehat{\mathrm{Box}}_E(0,r) = \big\{q \in \mathrm{Box}_E(0,r)\colon\ q = (0,\dots,0,x_\ell,\dots,x_s)\big\},$$

where $x_k = (x_{k1},\dots,x_{km_k}) \in \mathbb{R}^{m_k}$. By compactness and Lemma 3.6, there is a finite set $I \subset \widehat{\mathrm{Box}}_E(0,r)$ so that

$$\big\{\mathrm{Box}_{cc}\big(p, r^{\frac{1}{\ell-1}}\big)\colon\ p \in I\big\} \tag{3.8}$$

covers $\mathrm{Box}_E(0,r)$ and the elements of

$$\big\{\mathrm{Box}_{cc}\big(p, r^{\frac{1}{\ell-1}}/5C_{BB}^2\big)\colon\ p \in I\big\} \tag{3.9}$$

are pairwise disjoint.

Let us note that the union of the elements in the family appearing in (3.8) is in general a larger set than $\mathrm{Box}_E(0,r)$ and will be denoted by Ω.

If $p \in \widehat{\mathrm{Box}}_E(0,r)$ and $q \in \mathrm{Box}_{cc}(0, r^{\frac{1}{\ell-1}})$, then the Baker–Campbell–Hausdorff formula and an argument similar to the one in the proof of Lemma 3.3, show that there is a constant C_2 such that $p * q \in \tilde{\Omega}$ where

$$\tilde{\Omega} = \left[-r^{\frac{1}{\ell-1}}, r^{\frac{1}{\ell-1}}\right]^{m_1} \times \cdots \times \left[-r^{\frac{\ell-2}{\ell-1}}, r^{\frac{\ell-2}{\ell-1}}\right]^{m_{\ell-2}} \\ \times [-r,r]^{m_{\ell-1}} \times [-C_2 r, C_2 r]^{\sum_{j=\ell}^{s+1} m_j}.$$

It follows that $\mathrm{Box}_E(0,r) \subset \Omega \subset \tilde{\Omega}$, and since the family appearing in (3.9) is pairwise disjoint, we have

$$(\#I)\frac{2^N r^{\frac{Q}{\ell-1}}}{(5C_{BB}^2)^Q} \leqslant \mathrm{Vol}(\Omega) \leqslant \mathrm{Vol}(\tilde{\Omega}) = 2^N C_2^{\sum_{j=\ell}^{s+1} m_j} r^{\frac{1}{\ell-1}\sum_{j=0}^{\ell-2} j m_j + \sum_{j=\ell-1}^{s+1} m_j},$$

which implies

$$\#I \leqslant \left(5C_{BB}^2\right)^Q C_2^{\sum_{j=\ell}^{s+1} m_j} \frac{1}{r^{\lambda_+(\ell)}}. \tag{3.10}$$

Since $B_E(0,r) \subset \mathrm{Box}_E(0,r)$ the proof in the first stage is complete. Again, in the second stage of the proof we extend to the case of general centers. Let $B_E(p,r)$ be a closed ball with $0<r<1$ contained in K. Since $B_E(p,r)$ is compact, there is a finite set $J \subset B_E(p,r)$ so that

$$B_E(p,r) \subset \bigcup_{q\in J} B_E\left(q, r/C_{BB}^{\ell-1} C_1(K)\right),$$

where $\#J$ depends only on the constant $C_{BB}^{\ell-1} C_1(K)$ as follows from the $5r$ covering theorem and counting.

Using Lemma 3.3, (3.5), the result from the first stage and (3.6), it follows that

$$q^{-1} * B_E\left(q, r/C_{BB}^{\ell-1} C_1(K)\right) \subset B_E\left(0, r/C_{BB}^{\ell-1}\right) \subset \mathrm{Box}_E\left(0, r/C_{BB}^{\ell-1}\right) \\ \subset \bigcup_{p'\in I} \mathrm{Box}_{cc}\left(p', r^{\frac{1}{\ell-1}}/C_{BB}\right) \subset \bigcup_{p'\in I} B_{cc}\left(p', r^{\frac{1}{\ell-1}}\right),$$

hence

$$B_E\left(q, r/C_{BB}^{\ell-1} C_1(K)\right) \subset \bigcup_{p'\in I} B_{cc}\left(q * p', r^{\frac{1}{\ell-1}}\right)$$

and

$$B_E(p,r) \subset \bigcup_{q\in J}\bigcup_{p'\in I} B_{cc}\left(q * p', r^{\frac{1}{\ell-1}}\right).$$

Letting $M_+ = (\#J)(5C_{BB}^2)^Q C_2^{\sum_{j=\ell}^{s+1} m_j}$ and using (3.10) completes the proof. $\square$

Next we make preparations for the proof of Proposition 3.1. First we introduce the *α-dimensional spherical Hausdorff premeasure* of A which is defined in a similar way to the Hausdorff premeasure. It is given by

$$\mathcal{S}^{\alpha}_{(X,d),\delta}(A) = \inf \sum_{i=1}^{\infty} \operatorname{diam}\bigl(B(p_i, r_i)\bigr)^{\alpha},$$

where the infimum is taken over all coverings of A by metric balls $\{B(p_i, r_i)\}$ with diameter at most δ. For fixed α and A, the quantity $\mathcal{S}^{\alpha}_{(X,d),\delta}(A)$ is non-decreasing in δ and we let

$$\mathcal{S}^{\alpha}_{(X,d)}(A) = \mathcal{S}^{\alpha}_{(X,d),0}(A) := \sup_{\delta>0} \mathcal{S}^{\alpha}_{(X,d),\delta}(A)$$

be the *α-dimensional spherical Hausdorff measure* of A. The relationship between Hausdorff measure and spherical Hausdorff measure is summarized in the following proposition, see [47].

Proposition 3.7. *For each α, $\mathcal{H}^{\alpha}_{(X,d)}$ and $\mathcal{S}^{\alpha}_{(X,d)}$ are Borel regular (outer) measures on (X, d). Moreover,*

$$\mathcal{H}^{\alpha}_{(X,d)}(A) \leqslant \mathcal{S}^{\alpha}_{(X,d)}(A) \leqslant 2^{\alpha} \mathcal{H}^{\alpha}_{(X,d)}(A)$$

for all $A \subset X$.

Proposition 3.7 shows that up to a multiplicative constant, the same value is obtained if the Hausdorff measure $\mathcal{H}^{\alpha}_{(X,d)}$ is replaced by its spherical counterpart $\mathcal{S}^{\alpha}_{(X,d)}$. In particular, the associated notions of *Hausdorff dimension* and *spherical Hausdorff dimension* coincide. We replace the subscript (X, d) with E or cc when d is the Euclidean or Carnot–Carathéodory metric. We now commence the proof of Proposition 3.1.

Proof of Proposition 3.1. First we prove the existence of a constant $L_1 = L_1(\mathbb{G}, b)$ such that $\mathcal{H}^{\beta_+(\alpha)}_{cc}(S)/L_1 \leqslant \mathcal{H}^{\alpha}_{E}(S)$ for every $S \subset B_{cc}(0, b)$. Let $\mathcal{F}_E = \{B_E(p_i, r_i)\}_{i=1}^{\infty}$ be an arbitrary covering of S with Euclidean balls such that $0 < r_i < \delta/2 < 1$ and let $\ell \in \{2, \dots, s\}$; part (a) of Lemma 3.2 implies that

$$S \subset \bigcup_{i=1}^{\infty} B_E(p_i, r_i) \subset \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{n} B_{cc}\bigl(p_{ij}, r_i^{\frac{1}{\ell-1}}\bigr)$$

for a suitable family of CC balls $\{B_{cc}(p_{ij}, r_i^{\frac{1}{\ell-1}})\colon\ j = 1, \dots, n\}$, where

$$n \leqslant \frac{M_+}{r_i^{\lambda_+(\ell)}}$$

and $M_+ = M_+(\ell, B_{cc}(0, b))$. It follows that

$$\begin{aligned}
\mathcal{S}_{cc,\delta}^{(\ell-1)(\alpha+\lambda_+(\ell))}(S) &\leqslant \sum_{i=1}^{\infty}\sum_{j=1}^{n}\bigl(2r_i^{\frac{1}{\ell-1}}\bigr)^{(\ell-1)(\alpha+\lambda_+(\ell))} \\
&\leqslant 2^{(\ell-1)(\alpha+\lambda_+(\ell))}M_+\sum_{i=1}^{\infty} r_i^{\alpha} \\
&= 2^{(\ell-1)(\alpha+\lambda_+(\ell))-\alpha}M_+\sum_{i=1}^{\infty}\bigl(\operatorname{diam}_E B_E(p_i,r_i)\bigr)^{\alpha}.
\end{aligned}$$

Since $\mathcal{F}_E$ was arbitrary, we conclude that

$$\mathcal{S}_{cc,\delta}^{(\ell-1)(\alpha+\lambda_+(\ell))}(S) \leqslant 2^{(\ell-1)(\alpha+\lambda_+(\ell))-\alpha}M_+\mathcal{S}_{E,\delta}^{\alpha}(S).$$

Letting $\delta\to 0$, it follows that

$$\mathcal{S}_{cc}^{(\ell-1)(\alpha+\lambda_+(\ell))}(S) \leqslant 2^{(\ell-1)(\alpha+\lambda_+(\ell))-\alpha}M_+\mathcal{S}_{E}^{\alpha}(S),$$

and by Proposition 3.7 we have

$$\mathcal{H}_{cc}^{(\ell-1)(\alpha+\lambda_+(\ell))}(S) \leqslant 2^{(\ell-1)(\alpha+\lambda_+(\ell))}M_+\mathcal{H}_{E}^{\alpha}(S). \tag{3.11}$$

When $\ell=\ell_+$ is the value in (2.7) we have

$$\beta_+(\alpha) = (\ell-1)\bigl(\alpha+\lambda_+(\ell)\bigr), \tag{3.12}$$

and (3.11) becomes

$$\mathcal{H}_{cc}^{\beta_+(\alpha)}(S) \leqslant 2^{\beta_+(\alpha)}M_+\mathcal{H}_{E}^{\alpha}(S) \leqslant L_1\mathcal{H}_{E}^{\alpha}(S) \tag{3.13}$$

where $L_1 = 2^{Q}M_+$.

Next we prove the existence of a constant $L_2=L_2(\mathbb{G},b)$ such that $\mathcal{H}_E^{\alpha}(S)\leqslant L_2\mathcal{H}_{cc}^{\beta_-(\alpha)}(S)$ for every $S\subset B_{cc}(0,b)$. Let $\mathcal{F}_{cc}=\{B_{cc}(p_i,r_i)\}_{i=1}^{\infty}$ be an arbitrary covering of S with Carnot balls such that $0<r_i<\delta/2$ and let $\ell\in\{1,\dots,s-1\}$; Lemma 3.2 implies that

$$S\subset\bigcup_{i=1}^{\infty}B_{cc}(p_i,r_i)\subset\bigcup_{i=1}^{\infty}\bigcup_{j=1}^{n}B_E\bigl(p_{ij},r_i^{\ell+1}\bigr)$$

for a suitable family of Euclidean balls $\{B_E(p_{ij},r_i^{\ell+1})\colon\ j=1,\dots,n\}$, where

$$n\leqslant\frac{M_-(b)}{r_i^{\lambda_-(\ell)}}$$

and $M_-=M_-(\ell,B_{cc}(0,b))$. Since $\mathbb{G}$ is connected, $\operatorname{diam}_{cc}B_{cc}(p,r)\geqslant r$ for every $p\in\mathbb{G}$ and $r>0$, and

$$\mathcal{S}_{E,\delta}^{\alpha}(S)\leqslant\sum_{i=1}^{\infty}\sum_{j=1}^{n}\bigl(2r_i^{\ell+1}\bigr)^{\alpha}\leqslant 2^{\alpha}M_-\sum_{i=1}^{\infty}\bigl(\operatorname{diam}_{cc}B_{cc}(p_i,r_i)\bigr)^{(\ell+1)\alpha-\lambda_-(\ell)},$$

and since $\mathcal{F}_{cc}$ was arbitrary, we have

$$\mathcal{S}^{\alpha}_{E,\delta}(S) \leqslant 2^{\alpha} M_{-} \mathcal{S}^{(\ell+1)\alpha-\lambda_{-}(\ell)}_{cc,\delta}(S).$$

Letting $\delta \to 0$, it follows that

$$\mathcal{S}^{\alpha}_{E}(S) \leqslant 2^{\alpha} M_{-} \mathcal{S}^{(\ell+1)\alpha-\lambda_{-}(\ell)}_{cc}(S),$$

and by Proposition 3.7 we have

$$\mathcal{H}^{\alpha}_{E}(S) \leqslant 2^{\alpha+\alpha(\ell+1)-\lambda_{-}(\ell)} M_{-} \mathcal{H}^{(\ell+1)\alpha-\lambda_{-}(\ell)}_{cc}(S). \tag{3.14}$$

When $\ell = \ell_{-}$ is the value in (2.5) we have

$$\beta_{-}(\alpha) = (\ell+1)\alpha - \lambda_{-}(\ell), \tag{3.15}$$

and (3.14) becomes

$$\mathcal{H}^{\alpha}_{E}(S) \leqslant 2^{\alpha+\beta_{-}(\alpha)} M_{-} \mathcal{H}^{\beta_{-}(\alpha)}_{cc}(S) \leqslant L_2 \mathcal{H}^{\beta_{-}(\alpha)}_{cc}(S) \tag{3.16}$$

where $L_2 = M_{-} 2^{N+Q}$. Letting $L = \max\{2^Q M_{+}, 2^{N+Q} M_{-}\}$ and combining (3.13) with (3.16) completes the proof of Proposition 3.1. $\square$

The proofs of the box-counting dimension statements in Theorem 2.4 also use the Covering Lemma 3.2. We shall briefly indicate below a sketch of the proof for the box-counting dimension. The first step is to deduce from Lemma 3.2(a) an estimate of the form

$$N_{cc}\big(S, \epsilon^{\frac{1}{\ell-1}}\big) \leqslant \frac{M_{+}}{\epsilon^{\lambda_{+}(\ell)}} N_E(S, \epsilon)$$

for any bounded set $S \subset \mathbb{G}$, $\epsilon > 0$ and $\ell \in \{2, \dots, s-1\}$.

Using the above estimate it is easy to compute the upper and lower logarithmic rates of growth:

$$\frac{1}{\ell-1} \overline{\dim}^{B}_{cc}(S) \leqslant \overline{\dim}^{B}_{E}(S) + \lambda_{+}(\ell)$$

and

$$\frac{1}{\ell-1} \underline{\dim}^{B}_{cc}(S) \leqslant \underline{\dim}^{B}_{E}(S) + \lambda_{+}(\ell).$$

The right-hand inequality in (2.8) for upper/lower box counting dimension now follows by choosing $\ell = \ell_{+}$ and using (3.12) which gives

$$\overline{\dim}^{B}_{cc}(S) \leqslant \beta_{+}\big(\overline{\dim}^{B}_{E}(S)\big)$$

and

$$\overline{\dim}^{B}_{cc}(S) \leqslant \beta_{+}\big(\overline{\dim}^{B}_{E}(S)\big).$$

The proof of the left-hand inequality in (2.8) is similar, starting from an estimate of the form

$$N_E\big(S, \epsilon^{\ell+1}\big) \leqslant \frac{M_-}{\epsilon^{\lambda_-(\ell)}} N_{cc}(S, \epsilon).$$

We leave the details as an exercise to the reader.

4. Sharpness of the dimension comparison theorem

This section is divided into two parts. In the first part, we construct examples of vertical sets demonstrating sharpness of the upper dimension comparison function, while in the second (more complicated) part, we construct examples of horizontal sets demonstrating sharpness of the lower dimension comparison function.

Throughout this section and the next we make extensive use of the precise form of the group law in $\mathbb{G}$ as specified by the Baker–Campbell–Hausdorff formula. The key observation, which catalyzes our computations, is that the jth stratum expression in the group law is Euclidean in the jth stratum variable, sheared by polynomial maps in the lower strata variables. More precisely, $p * y = x$, where

$$x_j = p_j + y_j + \varphi_j(p_1, \dots, p_{j-1}, y_1, \dots, y_{j-1}) \tag{4.1}$$

and φ_j is a homogeneous polynomial with respect to the natural weights on the coordinates coming from the stratified structure of $\mathfrak{g}$. Here we used the representation of points in $\mathbb{G}$ in exponential coordinates: $p = (p_1, \dots, p_s)$, $p_j \in \mathbb{R}^{m_j}$. To simplify the numerous intricate expressions which occur, we introduce the following cumulative notation for the lowest strata variables:

$$P_j = (p_1, \dots, p_j) \in \mathbb{R}^{m_1+\dots+m_j}; \tag{4.2}$$

thus $p = P_s$ and (4.1) takes the form

$$x_j = p_j + y_j + \varphi_j(P_{j-1}, Y_{j-1}). \tag{4.3}$$

4.1. Vertical sets

In this subsection we prove the following theorem.

Theorem 4.1. *Let $\mathbb{G}$ be a Carnot group of step s with stratified Lie algebra $\mathfrak{g} = \mathfrak{v}_1 \oplus \dots \oplus \mathfrak{v}_s$. Let $m_j = \dim \mathfrak{v}_j$. For each $\ell = 1, \dots, s$ and each $\alpha \in [\sum_{j=\ell}^{s+1} m_j, \sum_{j=\ell-1}^{s+1} m_j]$ there exists a compact set $S \subset \mathbb{G}$ whose topological dimension is zero, such that*

$$\mathcal{H}_E^{\alpha}(S) < \infty \tag{4.4}$$

and

$$\mathcal{H}_{cc}^{\beta_+(\alpha)}(S) > 0. \tag{4.5}$$

Corollary 4.2. *The set S in Theorem* 4.1 *satisfies* $\dim_E S = \alpha$ *and* $\dim_{cc} S = \beta_+(\alpha)$.

Proof. (4.4) and (4.5) yield $\dim_E S \leqslant \alpha$ and $\dim_{cc} S \geqslant \beta_+(\alpha)$. Now use (2.8) and the strict monotonicity of β_+. $\square$

The main tool from geometric measure theory which we will use in the proof of Theorem 4.1 is the Mass Distribution Principle, see Theorem 8.7 and Definition 8.3 in [47] or Section 8.7 in [32].

Proposition 4.3 *(Mass Distribution Principle). Let μ be a positive measure on a metric space (X,d) so that $\mu(B(x,r)) \leqslant Cr^\beta$ for some constants C, β and all $r > 0$ and $x \in X$. Then $\mathcal{H}^\beta(X) > 0$.*

For each $m \in \mathbb{N}$ and each $0 \leqslant t \leqslant m$, let $C_t^m \subset \mathbb{R}^m$ be a compact set whose topological dimension is zero, whose Hausdorff and box-counting dimensions coincide and equal t, and which satisfies $0 < \mathcal{H}^t(C_t^m) < \infty$. See, e.g., Section 4.12 in [47] for a construction of such a set. When $t = 0$ we may choose $C_0^m = \{0\}$, while when $0 < t < m$, we may choose C_t^m to be a regular self-similar Cantor set of dimension t.

Next, we employ Frostman's lemma [47, Theorem 8.8] to choose a Borel probability measure μ_t on C_t^m satisfying the upper volume growth condition

$$\mu_t\big(C_t^m \cap \operatorname{Box}_E(p, R)\big) \leqslant K R^t \tag{4.6}$$

for all $p \in C_t^m$ and all $0 < R \leqslant \operatorname{diam}_E C_t^m$, for some fixed constant $K < \infty$. (The constant K may depend on m and t; this will have no effect on the argument which follows and we will suppress such dependence in the notation.)

In the proof of Theorem 4.1 we will use the following estimate for the Hausdorff measure of a product set. The statement and its proof are simple modifications of well-known estimates for the Hausdorff dimension of product sets, see for example [47, Theorem 8.10].

Lemma 4.4. *Let $A \subset \mathbb{R}^p$, $B \subset \mathbb{R}^q$ with $\mathcal{H}^a(A) < \infty$ and $\overline{\mathcal{M}}^b(B) < \infty$. Then $\mathcal{H}^{a+b}(A \times B) < \infty$. In particular,* $\dim^H(A \times B) \leqslant \dim^H(A) + \overline{\dim}^B(B)$.

Proof of Theorem 4.1. Intuitively, the statement of this theorem is obvious: a typical set $S \subset \mathbb{G}$ which is oriented in the direction of the higher strata as much as possible and with Euclidean dimension α should have CC dimension $\beta_+(\alpha)$.

We give the example in the form of a Euclidean product set and use Lemma 4.4 and the Mass Distribution Principle to establish (4.4) and (4.5). Without loss of generality we assume that $\alpha > m_s$, otherwise, choose a suitable Cantor set contained in the exponential of the highest stratum, $\exp(\mathfrak{v}_s)$. The example $S \subset \mathbb{G}$ will be the following (Euclidean self-similar) product set:

$$S = C_0^{m_1} \times \cdots \times C_0^{m_{\ell-2}} \times C_t^{m_{\ell-1}} \times C_{m_\ell}^{m_\ell} \times \cdots \times C_{m_s}^{m_s}, \tag{4.7}$$

where $t = \alpha - \sum_{j=\ell}^{s+1} m_j$. Clearly S is compact. The Product Theorem for topological dimension [36, Theorem III.4] implies that S has topological dimension zero.

We equip S with the probability measure

$$\mu = \mu_0 \times \cdots \times \mu_0 \times \mu_t \times \mu_{m_\ell} \times \cdots \times \mu_{m_s}.$$

When $t = 0$ or $t = m$ we have $\overline{\mathcal{M}}^t(C_t^m) < \infty$. For $t = 0$ this is trivial since the Minkowski content $\overline{\mathcal{M}}^0$ coincides with the counting measure. For $t = m$ the result follows since the Minkowski content $\overline{\mathcal{M}}^m$ on $\mathbb{R}^m$ is a multiple of Lebesgue measure. By Lemma 4.4, we conclude that (4.4) holds.

We now turn to the proof of (4.5). By the Mass Distribution Principle, it suffices to prove the volume growth estimate

$$\mu\big(S \cap B_{cc}(p,r)\big) \leqslant C r^{\beta_+(\alpha)} \tag{4.8}$$

for all p and r, with some absolute constant C. By the Ball–Box Theorem, (4.8) is equivalent with

$$\mu\big(S \cap \mathrm{Box}_{cc}(p,r)\big) \leqslant C r^{\beta_+(\alpha)}. \tag{4.9}$$

We expand the left-hand side of (4.9) as an iterated integral of the characteristic function of $S \cap \mathrm{Box}_{cc}(p,r)$:

$$\begin{aligned}\mu\big(S \cap \mathrm{Box}_{cc}(p,r)\big) = \int\limits_{C_0^{m_1}} d\mu_0(x_1) \cdots \int\limits_{C_0^{m_{\ell-2}}} d\mu_0(x_{\ell-2}) \int\limits_{C_t^{m_{\ell-1}}} d\mu_t(x_{\ell-1}) \\ \times \int\limits_{C_{m_\ell}^{m_\ell}} d\mu_{m_\ell}(x_\ell) \cdots \int\limits_{C_{m_s}^{m_s}} d\mu_{m_s}(x_s)\, \chi_{S\cap \mathrm{Box}_{cc}(p,r)}(x),\end{aligned} \tag{4.10}$$

where $x = (x_1, \ldots, x_s)$, $x_j \in \mathbb{R}^{m_j}$, is the representation of $x \in \mathbb{G}$ in exponential coordinates.

Next, we describe the structure of $S \cap \mathrm{Box}_{cc}(p,r)$. It is clear that $x \in \mathrm{Box}_{cc}(p,r)$ if and only if there exists $y = (y_1, \ldots, y_s)$ so that $|y_j| \leqslant r^j$ and (4.3) holds for all $j = 1, \ldots, s$. On the other hand, $x \in S$ if and only if $x_1 = 0, \ldots, x_{\ell-2} = 0$, $x_{\ell-1} \in C_t^{m_{\ell-1}}$, and $x_\ell \in [0,1]^{m_\ell}, \ldots, x_s \in [0,1]^{m_s}$. Consequently $x \in S \cap \mathrm{Box}_{cc}(p,r)$ if and only if

$$\begin{aligned} x_1 = p_1 + y_1 = 0, &\quad |y_1| \leqslant r, \\ x_2 = p_2 + y_2 + \varphi_2(p_1, y_1) = 0, &\quad |y_2| \leqslant r^2, \\ &\vdots \\ x_{\ell-2} = p_{\ell-2} + y_{\ell-2} + \varphi_{\ell-2}(P_{\ell-3}, Y_{\ell-3}) = 0, &\quad |y_{\ell-2}| \leqslant r^{\ell-2}, \\ x_{\ell-1} = p_{\ell-1} + y_{\ell-1} + \varphi_{\ell-1}(P_{\ell-2}, Y_{\ell-2}) \in C_t^{m_{\ell-1}}, &\quad |y_{\ell-1}| \leqslant r^{\ell-1}, \\ x_\ell = p_\ell + y_\ell + \varphi_\ell(P_{\ell-1}, Y_{\ell-1}) \in [0,1]^{m_\ell}, &\quad |y_\ell| \leqslant r^\ell, \\ &\vdots \\ x_s = p_s + y_s + \varphi_s(P_{s-1}, Y_{s-1}) \in [0,1]^{m_s}, &\quad |y_s| \leqslant r^s. \end{aligned} \tag{4.11}$$

Using (4.11), we define functions Ψ_j, $j = 1, \ldots, s$, inductively so that

$$y_j = \Psi_j(P_j, Y_{j-1}). \tag{4.12}$$

Observe that the first $\ell-2$ identities in (4.11) imply that $Y_{\ell-2}=(y_1,\dots,y_{\ell-2})$ is the vector consisting of the first $\ell-2$ coordinates of $q:=p^{-1}$, i.e., $\Psi_j(P_j,Y_{j-1})=q_j$ for $j=1,\dots,\ell-2$. Compare (4.1). Consequently, $\varphi_{\ell-1}(P_{\ell-2},Y_{\ell-2})=0$.

It follows that the characteristic function of the set $S\cap \mathrm{Box}_{cc}(p,r)$ is equal to the product of the following characteristic functions:

$$\begin{aligned}
h_{\ell-1}(x_{\ell-1}) &:= \chi_{\{x_{\ell-1}\in p_{\ell-1}+[-r^{\ell-1},r^{\ell-1}]^{m_{\ell-1}}\}}(x_{\ell-1}),\\
h_{\ell}(x_{\ell-1},x_\ell) &:= \chi_{\{x_\ell\in p_\ell+\varphi_\ell(P_{\ell-1},Y_{\ell-1})+[-r^{\ell},r^{\ell}]^{m_\ell}\}}(x_{\ell-1},x_\ell),\\
&\;\;\vdots\\
h_s(x_{\ell-1},\dots,x_s) &:= \chi_{\{x_s\in p_s+\varphi_s(P_{s-1},Y_{s-1})+[-r^{s},r^{s}]^{m_s}\}}(x_{\ell-1},\dots,x_s),
\end{aligned}$$

where the expressions $Y_{\ell-1},Y_\ell,\dots,Y_{s-1}$ are given recursively by (4.12), and $Y_j=Q_j$ for $j=1,\dots,\ell-2$.

We now return to (4.10) which we rewrite in the form

$$\int_{C_t^{m_{\ell-1}}} h_{\ell-1}(x_{\ell-1})\,d\mu_t(x_{\ell-1})\int_{C_{m_\ell}^{m_\ell}} h_\ell(x_{\ell-1},x_\ell)\,d\mu_{m_\ell}(x_\ell)\cdots\int_{C_{m_s}^{m_s}} h_s(x_{\ell-1},\dots,x_s)\,d\mu_{m_s}(x_s).$$

Estimating each integral in turn by starting from the last one and using (4.6), we find

$$\int_{C_{m_s}^{m_s}} h_s(x_{\ell-1},\dots,x_s)\,d\mu_{m_s}(x_s)=\mu_{m_s}\bigl(C_{m_s}^{m_s}\cap \mathrm{Box}_E\bigl(p_s+\varphi_s(P_{s-1},Y_{s-1}),r^s\bigr)\bigr)\leqslant Kr^{sm_s},$$

$$\begin{aligned}
&\int_{C_{m_{s-1}}^{m_{s-1}}} h_{s-1}(x_{\ell-1},\dots,x_{s-1})\,d\mu_{m_{s-1}}(x_{s-1})\\
&\quad=\mu_{m_{s-1}}\bigl(C_{m_{s-1}}^{m_{s-1}}\cap \mathrm{Box}_E\bigl(p_{s-1}+\varphi_{s-1}(P_{s-2},Y_{s-2}),r^{s-1}\bigr)\bigr)\leqslant Kr^{(s-1)m_{s-1}},
\end{aligned}$$

and so on, through

$$\int_{C_{m_\ell}^{m_\ell}} h_\ell(x_{\ell-1},x_\ell)\,d\mu_{m_\ell}(x_\ell)=\mu_{m_\ell}\bigl(C_{m_\ell}^{m_\ell}\cap \mathrm{Box}_E\bigl(p_\ell+\varphi_\ell(P_{\ell-1},Y_{\ell-1}),r^\ell\bigr)\bigr)\leqslant Kr^{\ell m_\ell}$$

and

$$\int_{C_t^{m_{\ell-1}}} h_{\ell-1}(x_{\ell-1})\,d\mu_t(x_{\ell-1})=\mu_t\bigl(C_t^{m_{\ell-1}}\cap \mathrm{Box}_E\bigl(p_{\ell-1},r^{\ell-1}\bigr)\bigr)\leqslant Kr^{(\ell-1)t}.$$

Combining all of these estimates gives

$$\mu\bigl(S\cap \mathrm{Box}_{CC}(p,r)\bigr)\leqslant K^{s-\ell+2}r^{(\ell-1)t+\sum_{j=\ell}^{s+1}jm_j}=K^{s-\ell+2}r^{\beta_+(\alpha)}$$

as desired. This completes the proof. □

Remark 4.5. The set S in Theorem 4.1 has well-defined Euclidean and CC box-counting dimensions, and $\dim_{cc}^B S = \beta_+(\dim_E^B S)$. Indeed, as a Euclidean self-similar set, S necessarily satisfies $\dim_E^B S = \dim_E^H S = \alpha$. Moreover,

$$\underline{\dim}_{cc}^B S \geqslant \dim_{cc}^H S = \beta_+(\alpha) = \beta_+\big(\dim_E^B S\big) \geqslant \overline{\dim}_{cc}^B S$$

which shows that the CC box-counting dimension of S exists and equals $\beta_+(\alpha)$.

Remark 4.6. In the preceding argument we may choose the set C_t^m to have any prescribed topological dimension less than or equal to t. More precisely, we may take C_t^m to be the product of a cube in $\mathbb{R}^{[t]}$ and a Cantor set of dimension $t-[t]$ in $\mathbb{R}$, where $[t]$ denotes the greatest integer less than or equal to t. The product formula $\dim_{top}(A \times B) = \dim_{top} A + \dim_{top} B$ need not hold in general, even for compact spaces A and B (see the remark following Theorem III.4 in [36]). Nevertheless, the set S defined as in (4.7) has topological dimension $[\alpha]$. Thus examples of vertical sets $S_{\alpha,\beta}$ can be constructed with any prescribed topological dimension in $[0,\alpha]$.

Remark 4.7. By work of Magnani and Magnani–Vittone, additional examples of low codimension vertical sets are given by certain smooth submanifolds of $\mathbb{G}$. Note that $\beta_+(\alpha) = Q-(N-\alpha)$ in case $N-m_1 \leqslant \alpha \leqslant N$. Let Σ be a bounded C^1-smooth submanifold of $\mathbb{G}$ of dimension α. Theorem 2.16 of [45] asserts the $(Q-(N-\alpha))$-negligibility of the horizontal subset $C(\Sigma)$ of Σ, see Definition 2.10 in [45] for the definition of $C(\Sigma)$. Then Theorem 1.2 of [46] yields, by standard theorems on measure differentiation and estimates for the metric factor $\theta(\tau_\Sigma^d)$, that Σ has positive $\mathcal{H}_{cc}^{Q-N+\alpha}$ measure. Since $\mathcal{H}_E^\alpha(\Sigma) < \infty$, we see that such submanifolds Σ are also examples of vertical sets for such values of α. See Subsection 8.2 for further remarks.

4.2. Horizontal sets

In this subsection we prove the following theorem.

Theorem 4.8. *Let $\mathbb{G}$ be a Carnot group of step s with stratified Lie algebra $\mathfrak{g} = \mathfrak{v}_1 \oplus \cdots \oplus \mathfrak{v}_s$. Let $m_j = \dim \mathfrak{v}_j$. For each $\ell = 0, \ldots, s-1$ and each $\alpha \in [\sum_{j=0}^{\ell} m_j, \sum_{j=0}^{\ell+1} m_j]$ there exists a bounded Borel set $S \subset \mathbb{G}$ whose topological dimension is zero, such that* $\dim_E S = \alpha$ *and* $\dim_{cc} S = \beta_-(\alpha)$.

Remark 4.9. We do not know whether the example in Theorem 2.6 can be chosen to be compact. In the proof we obtain compact examples for a countable dense set of dimension pairs $(\alpha, \beta_-(\alpha))$; the remaining examples are obtained by a density argument which only yields Borel sets. As follows from the proof, we can obtain F_σ examples if we remove the assumption on topological dimension.

Remark 4.10. We also do not know whether an example as in Theorem 2.6 can be chosen with positive (resp. finite) Hausdorff measures in the appropriate metrics, analogous to (4.4) and (4.5). Again, our proof only yields such examples for a countable dense set of pairs $(\alpha, \beta_-(\alpha))$.

Before beginning the proof of Theorem 4.8, we recall some basic facts from the theory of iterated function systems and self-similar fractal geometry. Let (X, d) be a complete metric space. A map $F : X \to X$ is *Lipschitz* if there exists $L < \infty$ so that

$$d\bigl(F(x), F(y)\bigr) \leqslant Ld(x, y) \tag{4.13}$$

for all $x, y \in X$. The infimum of all possible constants L which verify (4.13) is the *Lipschitz constant* of F, denoted $\mathrm{Lip}(F)$. (Subsequently we shall use the notation $\mathrm{Lip}_E(F)$, resp. $\mathrm{Lip}_{cc}(F)$, for the Lipschitz constant of a map F with respect to the Euclidean, resp. CC metric.) We say that F is *contractive* if $\mathrm{Lip}(F) < 1$. An *iterated function system* (IFS) on (X, d) is a finite collection $\mathcal{F}$ of contractive maps. To any IFS $\mathcal{F}$ there corresponds an *invariant set*, which is characterized as the unique nonempty compact set fully invariant under the action of $\mathcal{F}$. More precisely, the invariant set K for an IFS $\mathcal{F}$ satisfies

$$K = \bigcup_{f \in \mathcal{F}} f(K).$$

The existence and uniqueness of K follow from an application of a suitable fixed point theorem on the hyperspace of compact subsets of X equipped with the Hausdorff metric.

A map $f : X \to X$ is a *similarity* if there exists $r > 0$ so that $d(f(x), f(y)) = rd(x, y)$ for all $x, y \in X$. When $r < 1$ the map is contractive and we call r the *contraction ratio*. An IFS is *self-similar* if it is comprised of contractive similarities. The *similarity dimension* of such an IFS $\mathcal{F} = \{f_1, \ldots, f_M\}$ is the unique nonnegative solution t to the equation

$$\sum_{i=1}^{M} r_i^t = 1, \tag{4.14}$$

where r_i denotes the contraction ratio for f_i. An IFS $\mathcal{F} = \{f_1, \ldots, f_M\}$ satisfies the *open set condition* if there exists a nonempty bounded open set O so that the sets $f_i(O)$ are pairwise disjoint subsets of O. The following theorem is a standard tool in Euclidean self-similar fractal geometry, see Hutchinson [37], Kigami [39], or Falconer [21]. In the setting of doubling metric spaces, see [6].

Theorem 4.11. *Let (X, d) be a doubling metric space. Then the Hausdorff dimension of the invariant set K of any self-similar IFS in X is always less than or equal to the similarity dimension t, more precisely, $\mathcal{H}^t(K)$ is finite. Furthermore, equality between the Hausdorff, box-counting and similarity dimensions hold if the open set condition is satisfied. Indeed, if $\mathcal{F}$ is a self-similar IFS satisfying the open set condition, then $0 < \mathcal{H}^t(K) < \infty$ and*

$$\dim^H_{(X,d)} K = \dim^B_{(X,d)} K = t.$$

For our purposes, it suffices to note that Carnot groups equipped with the CC metric satisfy the doubling condition. In the first stage of our proof it will be crucial to relate an IFS in $\mathbb{G}$ with a corresponding IFS in the Euclidean space $\mathbb{R}^{m_1}$ which represents the first stratum in the stratification of $\mathfrak{g}$. We say that a map $F : \mathbb{G} \to \mathbb{G}$ *lifts* $f : \mathbb{R}^{m_1} \to \mathbb{R}^{m_1}$ if $\pi_1 \circ F = f \circ \pi_1$, where we recall that $\pi_1 : \mathbb{G} \to \mathbb{R}^{m_1}$ denotes projection to the first stratum. An IFS $F_1, \ldots, F_M$ on $\mathbb{G}$

lifts an IFS $f_1, \dots, f_M$ on $\mathbb{R}^{m_1}$ if F_i lifts f_i for each i, $i = 1, \dots, M$. A basic relation between Euclidean Lipschitz maps and their lifts which we will use is the following:

Lemma 4.12. *Let $F : \mathbb{G} \to \mathbb{G}$ be a contractive map which lifts $f : \mathbb{R}^{m_1} \to \mathbb{R}^{m_1}$. Then f is contractive, and $\mathrm{Lip}_E(f) \leqslant \mathrm{Lip}_{cc}(F)$. If $F(p) = q * \delta_r(p)$ is a contractive similarity, then f is a Euclidean similarity with the same contraction ratio $r > 0$.*

Proof. The first statement follows directly from (2.2) (and the subsequent statement regarding the case of equality) and (4.3). Let us note here that the inequality $\mathrm{Lip}_E(F) \leqslant \mathrm{Lip}_{cc}(F)$ is not true in general. The second statement follows directly from the explicit formulae of F and f for the case of similarities. □

A first step towards the proof of Theorem 4.8 is Proposition 2.7 which proves the theorem in the range $0 < \alpha \leqslant m_1$.

Proof of Proposition 2.7. Let $\{F_1, \dots, F_M\}$ and $\{f_1, \dots, f_M\}$ be as in the statement of the proposition, and let K be the invariant set for $\{F_1, \dots, F_M\}$. Then $\pi_1(K)$ is the invariant set for the (Euclidean self-similar) system $\{f_1, \dots, f_M\}$ on $\mathbb{R}^{m_1}$.

Since $\{f_1, \dots, f_M\}$ satisfies the open set condition in $\mathbb{R}^{m_1}$ we have by Theorem 4.11

$$0 < \mathcal{H}_E^\alpha\big(\pi_1(K)\big) < \infty,$$

where α is the similarity dimension of $\{f_1, \dots, f_M\}$. By Lemma 4.12 it follows that the similarity dimension of $\{F_1, \dots, F_M\}$ is also α. By the first part of Theorem 4.11, $\mathcal{H}_{cc}^\alpha(K) < \infty$. Now Proposition 3.1, specifically, the right-hand inequality in (3.1) implies

$$0 < \mathcal{H}_E^\alpha\big(\pi_1(K)\big) \leqslant \mathcal{H}_E^\alpha(K) \leqslant L\mathcal{H}_{cc}^\alpha(K) < \infty.$$

This completes the proof. □

In order to prove a generalization of Proposition 2.7 to higher strata we will make essential use the following integral estimate for the Hausdorff measures of level sets of a Lipschitz map. See Theorem 7.7 in [47].

Proposition 4.13. *Let $K \subset \mathbb{R}^n$, let $f : K \to \mathbb{R}^m$ be a Lipschitz map, and let $m \leqslant t \leqslant n$. If K is $\mathcal{H}^t$ measurable with $\mathcal{H}^t(K) < \infty$, then $\int \mathcal{H}^{t-m}(K \cap f^{-1}\{y\})\, d\mathcal{L}^m(y)$ exists and*

$$\int \mathcal{H}^{t-m}\big(K \cap f^{-1}\{y\}\big)\, d\mathcal{L}^m(y) \leqslant C\mathcal{H}^t(K),$$

where C depends only on m and the Lipschitz constant of f.

We will deduce Theorem 4.8 from the following proposition. Here we denote by

$$\Pi_\ell = \pi_1 \times \cdots \times \pi_\ell : \mathbb{G} \to \mathbb{R}^{\sum_{j=0}^{\ell} m_j}$$

the cumulative projection to the lowest ℓ strata.

Proposition 4.14. *Let $\mathbb{G}$, ℓ be as in Theorem* 4.8, *$b \geqslant 2$ an integer, and $M \in \{1, 2, \ldots, b^{(\ell+1)m_{\ell+1}}\}$. For each $j = 1, \ldots, s$, let $A_j = \{0, \ldots, b^j - 1\}^{m_j} \subset \mathbb{R}^{m_j}$. For $a_1 \in A_1, \ldots, a_k \in A_k$, let*

$$p_{a_1\cdots a_k} = (a_1, \ldots, a_k, 0, \ldots, 0)$$

and

$$F_{a_1\cdots a_k}(p) = p_{a_1\cdots a_k} * \delta_{1/b}\big(p^{-1}_{a_1\cdots a_k} * p\big).$$

Finally, let B be any subset of $A_{\ell+1}$ of cardinality M, let

$$\mathcal{F} = \{F_{a_1\cdots a_{\ell+1}}\colon\ a_1 \in A_1, \ldots, a_\ell \in A_\ell,\ a_{\ell+1} \in B\},$$

and let K be the invariant set for the CC self-similar IFS $\mathcal{F}$. Then

$$\mathcal{H}_{cc}^{\sum_{j=0}^{\ell} jm_j + \frac{\log M}{\log b}}(K) < \infty \tag{4.15}$$

and

$$\mathcal{H}_{E}^{\sum_{j=0}^{\ell} m_j + \frac{\log M}{\log b^{\ell+1}}}\big(\Pi_{\ell+1}(K)\big) > 0. \tag{4.16}$$

Moreover, if $M = b^{(\ell+1)m_{\ell+1}}$ (in which case $B = A_{\ell+1}$), then $\mathcal{H}_E^{\sum_{j=0}^{\ell+1} m_j}$-a.e. point $X_{\ell+1} \in \Pi_{\ell+1}(K)$ has a unique symbolic representation

$$X_{\ell+1} = \Pi_{\ell+1}\Big(\lim_{n\to\infty} F_{a_1^1\cdots a_{\ell+1}^1} \circ \cdots \circ F_{a_1^n\cdots a_{\ell+1}^n}(o)\Big) \tag{4.17}$$

for some unique symbol sequence

$$\sigma = \sigma(X_{\ell+1}) = \big\{\big(\big(a_1^1, \ldots, a_{\ell+1}^1\big), \big(a_1^2, \ldots, a_{\ell+1}^2\big), \ldots,\big)\big\} \in (A_1 \times \cdots \times A_{\ell+1})^{\mathbb{N}}.$$

To simplify notation in what follows, we write

$$F_\sigma(o) = \lim_{n\to\infty} F_{a_1^1\cdots a_{\ell+1}^1} \circ \cdots \circ F_{a_1^n\cdots a_{\ell+1}^n}(o)$$

so that (4.17) reads

$$X_{\ell+1} = \Pi_{\ell+1}\big(F_\sigma(o)\big). \tag{4.18}$$

Observe also that if

$$\alpha = \sum_{j=0}^{\ell} m_j + \frac{\log M}{\log b^{\ell+1}} \in \left[\sum_{j=0}^{\ell} m_j, \sum_{j=0}^{\ell+1} m_j\right] \tag{4.19}$$

is the exponent in (4.16), then

$$\beta_-(\alpha) = \sum_{j=0}^{\ell} j m_j + \frac{\log M}{\log b} \tag{4.20}$$

is the exponent in (4.15).

Proof of Theorem 4.8. Since $\Pi_{\ell+1} : (\mathbb{G}, d_E) \to (\mathbb{R}^{\sum_{j=0}^{\ell} m_j}, d_E)$ is Lipschitz, Eqs. (4.15) and (4.16) guarantee the existence of a compact set S satisfying

$$\mathcal{H}_E^{\alpha}(S) > 0 \tag{4.21}$$

and

$$\mathcal{H}_{cc}^{\beta_-(\alpha)}(S) < \infty. \tag{4.22}$$

in case α is of the form (4.19). An application of (2.8) together with the strict monotonicity of β_- completes the proof in this case. The set of all such α, as $b \geqslant 2$ and $M \in \{1, 2, \ldots, b^{(\ell+1)m_{\ell+1}}\}$ vary, is dense in the interval $[\sum_{j=0}^{\ell} m_j, \sum_{j=0}^{\ell+1} m_j]$. The case of general α follows from this and the monotonicity and countable stability of the Hausdorff dimension.

The set S constructed as in the previous paragraph need not have topological dimension zero. However, since it necessarily has finite topological dimension (in fact, $\dim_{top} S \leqslant N$), we may appeal to the decomposition theorem for topological dimension [36, Theorem III.3] to write S as the union of a finite number of Borel subsets, each of topological dimension zero. Replacing S by an appropriately chosen one of these subsets yields a bounded Borel set of topological dimension zero satisfying $\dim_E S = \alpha$ and $\dim_{cc} S = \beta_-(\alpha)$. $\square$

Remark 4.15. For α as in (4.19) we can find a compact set S of topological dimension zero satisfying (4.21) and (4.22). Indeed, for such α the subset described in the final sentence of the proof can be chosen satisfying (4.21) and (4.22). By a theorem of Howroyd (see [47, Chapter 8]) there exists a further compact subset (necessarily of topological dimension zero) which also satisfies (4.21) and (4.22).

Proof of Proposition 4.14. The proof will be by induction on ℓ.

Consider first the base case $\ell = 0$. Let $b \geqslant 2$ and $M \in \{1, \ldots, b^{m_1}\}$, let

$$A_1 = \{0, \ldots, b-1\}^{m_1} \subset \mathbb{R}^{m_1},$$
$$p_{a_1} = (a_1, 0, \ldots, 0) \in \mathbb{G}, \quad a_1 \in A_1,$$

and consider the contractive similarity of $(\mathbb{G}, d_{cc})$ given by

$$F_{a_1}(p) = p_{a_1} * \delta_{1/b}\big(p_{a_1}^{-1} * p\big).$$

Let $B \subset A_1$ be any set of cardinality M. The CC self-similar IFS $\mathcal{F} = \{F_a\colon a \in B\}$ has similarity dimension $\alpha = \log M / \log b$. Hence $\mathcal{H}_{cc}^{\alpha}(K) < \infty$ for the invariant set K. On the other hand,

$$\mathcal{H}_E^{\alpha}(K) \geqslant \mathcal{H}_E^{\alpha}\big(\pi_1(K)\big)$$

and $\pi_1(K)$ is the invariant set for the Euclidean self-similar IFS $\mathcal{F}_1 = \{f_a\colon a \in B\}$, $f_a(x) = a + \frac{1}{b}(x-a)$, on $\mathbb{R}^{m_1}$, which satisfies the open set condition with open set $O_1 = (0, b-1)^{m_1}$. Thus

$$\mathcal{H}_E^{\alpha}(K) > 0$$

as desired. If $M = b^{m_1}$ then $\mathcal{H}_E^{m_1}$-a.e. $x_1 \in \Pi_1(K) = \pi_1(K)$ has a unique symbolic representation relative to the IFS $\mathcal{F}$ (this is a consequence of the open set condition). This completes the proof in the case $\ell = 0$.

Now assume that the statement in the proposition is true for some integer $\ell - 1$ and all integers $b_0 \geqslant 2$ and $M_0 \in \{1, 2, \dots, b^{\ell m_\ell}\}$; we will prove that it holds true for ℓ and any given pair of integers $b \geqslant 2$ and $M \in \{1, 2, \dots, b^{(\ell+1)m_{\ell+1}}\}$. Let b and M be given. According to the inductive hypothesis in the $(\ell-1)$st step with $b_0 = b$ and $M_0 = b_0^{\ell m_\ell}$, the invariant set K_0 for the CC self-similar IFS

$$\mathcal{F}_0 = \{F_{a_1 \cdots a_\ell}\colon a_1 \in A_1, \dots, a_\ell \in A_\ell\},$$

satisfies the estimates

$$\mathcal{H}_{cc}^{\sum_{j=0}^{\ell} j m_j}(K_0) < \infty \tag{4.23}$$

and

$$\mathcal{H}_E^{\sum_{j=0}^{\ell} m_j}\big(\Pi_l(K_0)\big) > 0, \tag{4.24}$$

furthermore, almost every point $X_\ell \in \Pi_\ell(K_0)$ has a unique symbolic representation.

Now let B be any subset of $A_{\ell+1}$ of cardinality M, let $\mathcal{F}$ be the CC self-similar IFS comprised of the mappings $F_{a_1 \cdots a_{\ell+1}}$ for $a_1 \in A_1, \dots, a_\ell \in A_\ell$ and $a_{\ell+1} \in B$, and let K be the invariant set for $\mathcal{F}$. Note that

$$\Pi_\ell(K_0) \subseteq \Pi_\ell(K). \tag{4.25}$$

We will prove that (4.15) and (4.16) hold. The former follows immediately from the fact that $\mathcal{F}$ is CC self-similar with similarity dimension

$$\frac{\log(b^{\sum_{j=0}^{\ell} j m_j} M)}{\log b} = \beta_-(\alpha);$$

see (4.20).

To prove the latter, we will apply Proposition 4.13 with $t = \alpha$ as in (4.19), $m = \sum_{j=0}^{l} m_j$ and $f = \Pi_\ell$. We have to show that there exists a constant $c > 0$ so that

$$\mathcal{H}_E^{\alpha - \sum_{j=0}^{\ell} m_j}\big(K \cap \Pi_\ell^{-1}(X_\ell)\big) \geqslant c \tag{4.26}$$

for almost every $X_l \in \Pi_l(K)$.

In view of (4.24) and (4.25), Proposition 4.13 yields (4.22).

To prove (4.26) we begin with a lemma.

Lemma 4.16. *Let $\pi_q : \mathbb{G} \to \mathbb{R}^{m_q}$ denote projection to the qth stratum, $q = \ell + 1$. For every $X_\ell \in \Pi_\ell(K)$ which has a unique symbolic representation, the set $\pi_q(K \cap \Pi_\ell^{-1}(X_\ell))$ is a Euclidean translate of the invariant set $K' \subset \mathbb{R}^{m_q}$ of the Euclidean self-similar IFS*

$$\mathcal{G} = \{g_{a_q}\colon a_q \in B\}, \tag{4.27}$$

where $g_a(x) = \frac{1}{b^q}x + (1 - \frac{1}{b^q})a$.

We emphasize that in Lemma 4.16 the translation parameter depends on $X_\ell = (x_1, \dots, x_\ell)$, but the IFS $\mathcal{G}$ does not.

Proof of Lemma 4.16. Let X_ℓ be as in the statement and let $\sigma = \sigma(X_\ell) \in (A_1 \times \cdots \times A_\ell)^{\mathbb{N}}$ be the unique associated symbol string. In the remainder of this proof, we use $x' = (x_1', \dots, x_s')$ for a dummy variable in $\mathbb{G}$, as well as our standard notation $X_\ell' = (x_1', \dots, x_\ell')$.

Let us first consider the expression

$$\pi_q \circ F_{a_1 \cdots a_q}(x') - \left[\frac{1}{b^q}x_q + \left(1 - \frac{1}{b^q}\right)a_q\right] \tag{4.28}$$

as a function of $x' \in K$ and $(a_1, \dots, a_q)$. From the form of the Baker–Campbell–Hausdorff formula (see especially (4.3)) we conclude that the expression in (4.28) is independent of x_q' and a_q, i.e., depends only on X_ℓ' and $(a_1, \dots, a_\ell)$. We write

$$\pi_q \circ F_{a_1 \cdots a_q}(x') = \frac{1}{b^q}x_q' + \left(1 - \frac{1}{b^q}\right)a_q + \Phi_1\big(X_\ell'; a_1, \dots, a_\ell\big) \tag{4.29}$$

for some real-valued function Φ_1 defined on $\Pi_\ell(K) \times (A_1 \times \cdots \times A_\ell)$.

Similarly, for any n, the expression

$$\pi_q \circ F_{a_1^1 \cdots a_q^1} \circ \cdots \circ F_{a_1^n \cdots a_q^n}(x) - \left[\sum_{m=1}^n \frac{1}{b^{q(m-1)}}\left(1 - \frac{1}{b^q}\right)a_q^m + \frac{1}{b^{qn}}x_q'\right], \tag{4.30}$$

considered as a function of $x' \in K$ and $(a_1^1, \dots, a_q^1), \dots, (a_1^n, \dots, a_q^n)$ is independent of x_q' and $a_q^1, \dots, a_q^n$, i.e., depends only on X_ℓ' and $(a_1^1, \dots, a_\ell^1), \dots, (a_1^n, \dots, a_\ell^n)$. We write

$$\begin{aligned} &\pi_q \circ F_{a_1^1 \cdots a_q^1} \circ \cdots \circ F_{a_1^n \cdots a_q^n}(x') \\ &\quad = \frac{1}{b^{qn}}x_q + \sum_{m=1}^n \frac{1}{b^{q(m-1)}}\left(1 - \frac{1}{b^q}\right)a_q^m + \Phi_n\big(X_\ell'; \big(a_1^1, \dots, a_\ell^1\big), \dots, \big(a_1^n, \dots, a_\ell^n\big)\big) \end{aligned} \tag{4.31}$$

for some real-valued function Φ_n defined on $\Pi_\ell(K) \times (A_1 \times \cdots \times A_\ell)^n$.

This behavior passes to the limit as $n \to \infty$, where we conclude that

$$\pi_q \circ F_\sigma(x') = \sum_{m=1}^\infty \frac{1}{b^{q(m-1)}}\left(1 - \frac{1}{b^q}\right)a_q^m + \Phi_\infty\big(X_\ell', \sigma\big) \tag{4.32}$$

for some real-valued function Φ_∞ defined on $\Pi_\ell(K) \times (A_1 \times \cdots \times A_\ell)^{\mathbb{N}}$. Since $\sigma = \sigma(X_\ell)$ is uniquely determined by X_ℓ, we can write (4.32) as

$$\pi_q \circ F_\sigma(x') = \sum_{m=1}^{\infty} \frac{1}{b^{q(m-1)}}\left(1 - \frac{1}{b^q}\right) a_q^m + \Phi_\infty\big(X'_\ell, \sigma(X_\ell)\big). \tag{4.33}$$

Next, we observe that x_q is in $\pi_q(K \cap \Pi_\ell^{-1}(X_\ell))$ if and only if there exists a symbol string $\sigma \in (A_1 \times \cdots \times A_\ell \times B)^{\mathbb{N}}$ so that $x_q = \pi_q(F_\sigma(o))$. By (4.33) this means

$$x_q = \sum_{m=1}^{\infty} \frac{1}{b^{q(m-1)}}\left(1 - \frac{1}{b^q}\right) a_q^m + \Phi_\infty\big((0, \ldots, 0), \sigma(X_\ell)\big) \tag{4.34}$$

for a suitable choice of $(a_q^1, a_q^2, \ldots) \in B^{\mathbb{N}}$. Set $R(X_\ell) = \Phi_\infty((0, \ldots, 0), \sigma(X_\ell))$. Eq. (4.34) holds for some sequence $(a_q^1, a_q^2, \ldots) \in B^{\mathbb{N}}$ if and only if $x_q \in K' + R(X_\ell)$. Thus

$$\pi_q\big(K \cap \Pi_\ell^{-1}(X_\ell)\big) = K' + R(X_\ell).$$

This completes the proof of the lemma. $\square$

With this proof in hand we now quickly complete the proof of the proposition. The IFS $\mathcal{G}$ in (4.27) satisfies the open set condition (use the open set $O = (0, b^q - 1)^{m_q}$) and has similarity dimension $\log M / \log b^q = \alpha - \sum_{j=0}^{\ell} m_j$, see (4.19). Thus

$$\mathcal{H}_E^{\alpha - \sum_{j=0}^{\ell} m_j}\big(K \cap \Pi_\ell^{-1}(X_\ell)\big) \geqslant \mathcal{H}_E^{\alpha - \sum_{j=0}^{\ell} m_j}\big(\pi_q\big(K \cap \Pi_\ell^{-1}(X_\ell)\big)\big) = \mathcal{H}_E^{\log M / \log b^q}(K') > 0$$

for almost every $X_\ell \in \Pi_\ell(K)$. This completes the proof of (4.26) and hence also the proof of (4.16). Moreover, the identity in (4.34) shows that each point in $\Pi_q(K \cap \Pi_\ell^{-1}(X_\ell))$ has a unique symbolic representative, provided that X_ℓ and also x_q do. If $M = b^{q m_q}$, $\mathcal{H}_E^{\sum_{j=0}^{q} m_j}$-a.e. point in $\Pi_q(K)$ is of this type, by Fubini's theorem. This completes the proof of Proposition 4.14. $\square$

Remark 4.17. The set S in Theorem 4.8 has well-defined Euclidean and CC box-counting dimensions, and $\dim_{cc}^B S = \beta_-(\dim_E^B S)$. Indeed, as a CC self-similar set, S necessarily satisfies $\dim_{cc}^B S = \dim_{cc}^H S = \beta_-(\alpha)$. Moreover,

$$\overline{\dim}_E^B S \leqslant (\beta_-)^{-1}\big(\dim_{cc}^B S\big) = \alpha = \dim_E^H S \leqslant \underline{\dim}_E^B S$$

which shows that the Euclidean box-counting dimension of S exists and equals α.

Remark 4.18. We reiterate the purely Euclidean consequences of Theorem 4.8. The CC self-similar iterated function systems constructed in Proposition 4.14 can be viewed as iterated function systems in the underlying Euclidean geometry; in view of the nilpotence of $\mathbb{G}$ and the Baker–Campbell–Hausdorff formula the associated mappings are polynomial of an *a priori* high degree. Thus, viewed in Euclidean terms, these IFS are nonlinear and nonconformal; it is typically quite difficult to calculate explicitly the dimensions of such systems by traditional methods. Nevertheless, by our approach we obtain an explicit formula for their Euclidean Hausdorff dimension. As an illustration, we restate Example 2.10 in purely Euclidean terms. Consider the four maps of $\mathbb{R}^4$ given by

$$F_1(x) = \left(\frac{1}{2}x_1, \frac{1}{2}x_2, \frac{1}{4}x_3, \frac{1}{8}x_4\right), \tag{4.35}$$

$$F_2(x) = \left(\frac{1}{2}x_1 + \frac{1}{2}, \frac{1}{2}x_2, \frac{1}{4}x_3, \frac{1}{8}x_4\right), \tag{4.36}$$

$$F_3(x) = \left(\frac{1}{2}x_1, \frac{1}{2}x_2 + \frac{1}{2}, \frac{1}{4}x_3 + \frac{1}{4}x_1, \frac{1}{8}x_4 + \frac{3}{16}x_1^2\right), \tag{4.37}$$

and

$$F_4(x) = \left(\frac{1}{2}x_1 + \frac{1}{2}, \frac{1}{2}x_2 + \frac{1}{2}, \frac{1}{4}x_3 + \frac{1}{4}(x_1 - 1), \frac{1}{8}x_4 + \frac{1}{16}(x_1 - 1)^2\right), \tag{4.38}$$

where $x = (x_1, x_2, x_3, x_4)$. These are evidently not global contractive maps of $(\mathbb{R}^4, d_E)$. However, since they are precisely the contractive similarities of the Carnot Engel group specified in Example 2.2, we know that they generate a compact invariant set in $\mathbb{R}^4$ whose Euclidean Hausdorff dimension is exactly equal to 2. (See Section 2 for pictures of some three-dimensional projections of this set.)

Remark 4.19. We conclude this section by discussing the implications of Theorems 2.4 and 2.6 for the theory of codimension one horizontal rectifiability as developed by Franchi, Serapioni and Serra Cassano [27,26]. Let $\mathbb{G}$ be a Carnot group. We recall the following definitions from [26]:

(i) $u : \mathbb{G} \to \mathbb{R}$ is a $C^1_{\mathbb{G}}$ function if Xu is continuous for all $X \in V_1$,
(ii) the *horizontal gradient* of a $C^1_{\mathbb{G}}$ function $u : \mathbb{G} \to \mathbb{R}$ is the unique map $\nabla_{\mathbb{G}} u : \mathbb{G} \to V_1$ satisfying $Xu = \langle X, \nabla_{\mathbb{G}} u\rangle$ for all $X \in V_1$,
(iii) a codimension one hypersurface S is $\mathbb{G}$-*regular* if it is locally the zero set of a $C^1_{\mathbb{G}}$ function with nonvanishing horizontal gradient,
(iv) a set S in $\mathbb{G}$ is called *horizontally* $(Q-1)$-*rectifiable* (or *horizontally rectifiable in codimension one*) if S is the union of a countable family of $\mathbb{G}$-regular hypersurfaces, together with a set of $\mathcal{H}^{Q-1}_{cc}$-dimensional measure zero.

We say that $S \subset \mathbb{G}$ is k-*rectifiable*, $0 \leqslant k \leqslant N$, if it is rectifiable in the classical Euclidean sense as a subset of $\mathbb{R}^N$: S is the union of a countable family of Lipschitz images of subsets of $\mathbb{R}^k$, together with a set of $\mathcal{H}^k_E$-dimensional measure zero.

It is of interest to understand the difference between notions of Euclidean $(N-1)$-rectifiability and the horizontal codimension one rectifiability of subsets of $\mathbb{G}$ with underlying space $\mathbb{R}^N$. The following corollary to Theorem 2.6 extends [5, Theorem 5.1] to the setting of general Carnot groups.

Corollary 4.20. *Let $\mathbb{G}$ be a Carnot group of dimension N and homogeneous dimension Q. Then every $(N-1)$-rectifiable set in $\mathbb{G}$ is horizontally $(Q-1)$-rectifiable. In every nonabelian Carnot group, there exist horizontally $(Q-1)$-rectifiable sets $S \subset \mathbb{G}$ which are not $(N-1)$-rectifiable.*

Proof. Let $S \subset \mathbb{G} = \mathbb{R}^N$ be $(N-1)$-rectifiable. By standard Euclidean geometric measure theory, $S = Z \cup \bigcup_{i=1}^{\infty} S_i$, where S_i is the zero set of a C^1 function $f_i : \mathbb{R}^N \to \mathbb{R}$ and $\mathcal{H}_E^{N-1}(Z) = 0$. We denote by $C(S_i)$ the characteristic set of the hypersurface S_i, i.e., the set of points $x \in S_i$ for which $H_x\mathbb{G} \subset T_x S_i$. The complement of $C(S_i)$ in S_i is the subset of $\{f_i = 0\}$ on which $\nabla_{\mathbb{G}} f_i \neq 0$. By [43, Theorem 6.6.2], $\mathcal{H}_{cc}^{Q-1}(C(S_i)) = 0$. By Proposition 3.1, $\mathcal{H}_{cc}^{Q-1}(Z) = 0$. We have

$$S = \left(Z \cup \bigcup_{i=1}^{\infty} C(S_i) \right) \cup \bigcup_{i=1}^{\infty} \bigl(S_i \setminus C(S_i)\bigr) = Z' \cup \bigcup_{i=1}^{\infty} S_i',$$

where $\mathcal{H}_{cc}^{Q-1}(Z') = 0$ and S_i' is a $\mathbb{G}$-regular hypersurface. Thus S is horizontally $(Q-1)$-rectifiable.

To construct a set S in a nonabelian Carnot group $\mathbb{G}$ as in the second assertion, observe that $(\beta_-)^{-1}(Q-1) = N - s^{-1}$, where s is the step of the group. Since $\mathbb{G}$ is nonabelian, $s \geqslant 2$. We may choose a pair of monotone increasing sequences (α_ν) and (β_ν) satisfying $N - 1 < \alpha_1$, $\lim_{\nu\to\infty} \alpha_\nu = N - s^{-1}$, and $\beta_\nu = \beta_-(\alpha_\nu)$. With $S_{\alpha,\beta}$ the set constructed in Theorem 2.6, we have

$$S = \bigcup_{\nu=1}^{\infty} S_{\alpha_\nu,\beta_\nu}$$

satisfies $\mathcal{H}_{cc}^{Q-1}(S) = 0$ (so S is trivially horizontally $(Q-1)$-rectifiable) but $\dim_E S \geqslant \alpha_1 > N - 1$. In fact we have that $\dim_E S = N - s^{-1}$ and so S is not $(N-1)$-rectifiable. $\square$

Kirchheim and Serra Cassano [40] have constructed an $\mathbb{H}^1$-regular hypersurface in $\mathbb{H}^1$ whose Euclidean Hausdorff dimension is 2.5, even locally at every point. Note that $2.5 = (\beta_-^{\mathbb{H}^1})^{-1}(3)$. In any Carnot group $\mathbb{G}$, does there exist a $\mathbb{G}$-regular hypersurface of Euclidean dimension $N - s^{-1}$?

5. CC self-similar invariant sets in Carnot groups are almost surely horizontal

This section is devoted to the proof of Theorem 2.8, which establishes the equality

$$\dim_{cc} K = \beta_-(\dim_E K)$$

almost surely for generic members of certain finite-dimensional parameterized families of CC self-similar sets K in a Carnot group $\mathbb{G}$. Inspiration for this type of result comes from work of Falconer [20,22], which establishes similar results for generic members of certain families of self-affine invariant sets in Euclidean space. In view of the fact that the group operation in $\mathbb{G}$ is given by polynomial maps, our results return purely Euclidean dividends: we obtain almost sure dimension statements for families of nonlinear, nonconformal Euclidean invariant sets. In the following section, we illustrate this point in the jet space Carnot groups.

Let us begin by briefly recalling the work of Falconer [20]. The *singular value function* of a contractive linear map $A:\mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$\varphi^t(A) = \begin{cases} 1, & t = 0, \\ \mu_1\mu_2\cdots\mu_{m-1}\mu_m^{t-m+1}, & m-1 < t \leqslant m, \\ (\mu_1\cdots\mu_n)^{t/n}, & t \geqslant n, \end{cases}$$

where $1 > \mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n \geqslant 0$ denote the singular values of A, i.e., the positive square roots of the eigenvalues of A^*A. The *operator norm* of A is the largest eigenvalue $\|A\| = \mu_1$.

Next, let $\mathbf{A} = \{A_1, \dots, A_M\}$ be a finite collection of contractive linear maps. Then for any $t \geqslant 0$ the limit

$$\lim_{m\to\infty}\biggl(\sum_{w:\ |w|=k} \varphi^t(A_w)\biggr)^{1/m} \tag{5.1}$$

exists, where the sum is taken over all words $w = w_1w_2\cdots w_m$ of length m in the letters $\{1, 2, \dots, M\}$ and $A_w = A_{w_1}\cdots A_{w_m}$. (For a more complete review of the symbolic dynamics of iterated function systems, see Subsection 5.1.) The expression in (5.1) is a strictly decreasing, continuous function of t, and we let

$$\begin{aligned} d(\mathbf{A}) = {} & \text{the unique nonnegative value of } t \text{ such} \\ & \text{that the quantity in (5.1) is equal to one.} \end{aligned} \tag{5.2}$$

Falconer [20] proved Theorem 5.1 with $\|A_i\| < \frac{1}{3}$ for all i; the stated generalization is due to Solomyak [57].

Theorem 5.1 *(Falconer, Solomyak). Assume that $\|A_i\| < \frac{1}{2}$ for all $i = 1, \dots, M$. Then, for almost every $\mathbf{b} = (b_1, \dots, b_M) \in \mathbb{R}^{n\times M}$, we have*

$$\dim_E K(\mathbf{b}) = \min\bigl\{d(\mathbf{A}), n\bigr\},$$

where $K(\mathbf{b})$ denotes the invariant set for the affine IFS $\{F_1, \dots, F_M\}$, $F_i(x) = A_ix + b_i$.

Theorem 5.1 has been generalized to the setting of horizontal self-affine IFS in the first Heisenberg group by the first two authors in [7] (see also [4]). The purpose of this section is to prove Theorem 2.8 which provides a far-reaching generalization of Falconer's almost sure dimension result to the setting of horizontal self-similar IFS in general Carnot groups. A further generalization to horizontal self-affine Carnot IFS is presumably possible, but we do not address this here.

As will be explained in more detail in the following section, the transition formula

$$\alpha = (\beta_-)^{-1}(\beta)$$

in (2.10), which arises from the lower dimension comparison statement in Theorem 2.4, encodes the same information as Falconer's formulas (5.1) and (5.2) in the setting of jet space groups.

5.1. Symbolic dynamics and iterated function systems

In this subsection, we recall the formalism of symbolic dynamics in the context of iterated function systems. Our notation follows [39]. Let $\mathcal{F} = \{F_1, \dots, F_M\}$ be a self-similar IFS on a complete metric space (X, d), and denote by r_i the contraction ratio associated to F_i. For $k \geqslant 1$, define

$$W_k := \{1, \dots, M\}^k = \bigl\{w_1 \cdots w_k\colon\ w_m \in \{1, \dots, M\},\ 1 \leqslant m \leqslant k\bigr\},$$

called the set of *words of length k* in the alphabet $\{1, \dots, M\}$. For $k = 0$, set $W_0 = \{\emptyset\}$ and call $\emptyset$ the *empty word*. Finally, define

$$W_* = \bigcup_{k=0}^{\infty} W_k$$

to be the *set of finite sequences* and

$$\Sigma = \{w_1 w_2 \cdots\colon\ w_m \in W_1\}$$

the *set of infinite sequences*.

We write vw for the concatenation of two words $v, w \in W_*$: $vw = v_1 v_2 \cdots v_k w_1 w_2 \cdots w_l$ if $v = v_1 \cdots v_k \in W_k$ and $w = w_1 \cdots w_l \in W_l$. If $w = vv'$ for some word v' we say that v is a *subword* of w. The *largest common subword* of v and w will be denoted $v \wedge w$; this is characterized as the unique common subword of v and w which is maximal with respect to length. We will abuse notation slightly, denoting by w both finite and infinite words.

For $S \subset X$ and $w = w_1 \cdots w_k \in W_*$, define $F_w = F_{w_1} \circ \cdots \circ F_{w_k}$, $r_w = r_{w_1} \cdots r_{w_k}$, and $S_w = F_w(S)$.

We equip Σ with the product topology induced by the discrete topology on the alphabet. Then there exists a canonical continuous surjection $\pi : \Sigma \to K$ to the invariant set of $\mathcal{F}$, characterized by the relation $\{\pi(w)\} = \bigcap_{k=1}^{\infty} F_{w_1 \cdots w_k}(K)$. Alternatively,

$$\pi(w) = \lim_{k \to \infty} F_{w_1 \cdots w_k}(x_0)$$

for any fixed $x_0 \in X$. The map π is called the *canonical symbol map* for the IFS $\{F_1, \dots, F_M\}$.

We record the commutation relation

$$\pi \circ \sigma_w = F_w \circ \pi, \quad w \in W_*, \tag{5.3}$$

where $\sigma : \Sigma \to \Sigma$ denotes the left shift,

$$\sigma(w_1 w_2 w_3 \cdots) = w_2 w_3 \cdots,$$

and $\sigma_w : \Sigma \to \Sigma$ the map which prepends w to its argument,

$$\sigma_w(v) = wv.$$

The *cylinder set* over $w \in W_*$ is $\Sigma_w = \sigma_w(\Sigma)$; this set consists of all infinite words which begin with w. By (5.3), $\pi(\Sigma_w) = K_w$. A *partition* of Σ is a disjoint collection of cylinder sets which covers Σ.

5.2. Proof of Theorem 2.8

The proof of Theorem 2.8 uses energy estimates to obtain almost sure lower bounds on Hausdorff dimension. We recall the following standard result, see, for example Theorems 4.2 and 4.13 in [21] or Theorem 8.7 in [47].

Proposition 5.2. *Let S be a subset of a complete metric space (X, d) and let μ be a positive and finite Borel regular measure supported on S whose s-energy*

$$\int_X \int_X d(x, y)^{-s}\, d\mu(x)\, d\mu(y)$$

is finite. Then the Hausdorff dimension of S is at least s.

Assume that $\mathcal{F}$ is a self-similar IFS as above, and let $t \geqslant 0$ be the similarity dimension for $\mathcal{F}$. Following Kigami [39], we introduce a probability measure λ on the symbol space Σ as follows:

$$\lambda(E) = \lim_{m\to\infty} \inf_{\Lambda} \sum_{\substack{w\in\Lambda \\ E\cap\Sigma_w\neq\emptyset}} r_w^t, \tag{5.4}$$

where the infimum is taken over all partitions Λ of Σ into cylinder sets defined by words of length at least m. Note that

$$\lambda(\Sigma_w) = r_w^t \tag{5.5}$$

for cylinder sets Σ_w, indeed, $\sum_{v\in\Lambda:\ \Sigma_w\cap\Sigma_v\neq\emptyset} r_v^t = r_w^t$ if Λ partitions by words of length at least $|w|$.

Let us recall the notations for $\mathbf{r}$, $\mathbf{P}$, $\alpha(\mathbf{r})$, $\beta(\mathbf{r})$ introduced before the statement of Theorem 2.8. In our proofs below we shall consider the measure λ for the value $t = \beta(\mathbf{r})$ and we emphasize on the fact that λ depends only on $\mathbf{r}$ and not on $\mathbf{P}$. We will use λ in this connection for invariant sets $K(\mathbf{P})$ where $\mathbf{P}$ varies but $\mathbf{r}$ is fixed. Furthermore, for $\mathbf{P} \in \mathbb{G}^M$ we denote the canonical symbol map by $\pi_{\mathbf{P}}: \Sigma \to K(\mathbf{P})$. An essential ingredient in the proof of Theorem 2.8 is the following statement.

Proposition 5.3. *Let $\mathbb{G}$ and $\mathbf{r}$ be as in Theorem* 2.8*, and define $\alpha = \alpha(\mathbf{r})$ and $\beta = \beta(\mathbf{r})$ as before. For each $0 < R < \infty$ and $\alpha' < \alpha$,*

$$\int_{B(R)^M} \int_\Sigma \int_\Sigma \left|\pi_{\mathbf{P}}(u) - \pi_{\mathbf{P}}(v)\right|_E^{-\alpha'} d\lambda(u)\, d\lambda(v)\, d\mathbf{P} < \infty, \tag{5.6}$$

where $B(R)$ denotes the (Euclidean) ball of radius R in $\mathbb{G}$ centered at $o \in \mathbb{G}$, $d\mathbf{P}$ denotes the element of integration with respect to the M-fold product of Haar measures on $\mathbb{G}^M$, and $d\lambda$ is the measure defined in (5.4) *with $t = \beta$.*

Proof of Theorem 2.8. As discussed in Section 2, to prove the Hausdorff dimension statements in Theorem 2.8 it suffices to prove the inequality

$$\dim_E K(\mathbf{P}) \geqslant \alpha$$

for almost every $\mathbf{P} \in \mathbb{G}^M$. For each $0 < R < \infty$, we obtain from (5.6) that

$$\int_\Sigma \int_\Sigma \left|\pi_{\mathbf{P}}(u) - \pi_{\mathbf{P}}(v)\right|_E^{-\alpha'} d\lambda(u)\, d\lambda(v) < \infty$$

for almost every $\mathbf{P} \in B(R)^M$, hence

$$\int_{K(\mathbf{P})} \int_{K(\mathbf{P})} |p - q|_E^{-\alpha'} d\big((\pi_{\mathbf{P}})_\# \lambda\big)(p)\, d\big((\pi_{\mathbf{P}})_\# \lambda\big)(q) < \infty$$

where the integration is with respect to the pushforward measure $(\pi_{\mathbf{P}})_\# \lambda$. By Proposition 5.2, $\dim_E K(\mathbf{P}) \geqslant \alpha'$ for every such $\mathbf{P}$. Letting $\alpha' \to \alpha$ and $R \to \infty$ completes the proof in the Hausdorff dimension case. The box-counting dimension case follows once we observe that condition (a) for upper box-counting dimension holds by the general theory of iterated function systems. $\square$

We derive Proposition 5.3 from the following technical lemma which is at the heart of the proof.

Lemma 5.4. *Assume that $\gamma := \max\{r_1, \ldots, r_M\} < \frac{1}{2}$. For each $R < \infty$ and $0 \leqslant s \leqslant N$, there exists a constant $C = C(R, \mathbb{G}, \alpha, \gamma)$ so that*

$$\int_{B(R)^M} \left|\pi_{\mathbf{P}}(v)^{-1} * \pi_{\mathbf{P}}(u)\right|_E^{-\alpha} d\mathbf{P} \leqslant \frac{C}{r_{u \wedge v}^{\beta_-(\alpha)}} \tag{5.7}$$

for all $u, v \in \Sigma$.

Proof of Proposition 5.3. Let $\beta' = \beta_-(\alpha')$. Recall that β is defined as in (2.9). By monotonicity of β_-, $\beta > \beta'$. Using Fubini's theorem, (5.7), (3.3) and (5.5), we estimate the integral in (5.6) as follows:

$$\begin{aligned}
&\int_{B(R)^M} \int_\Sigma \int_\Sigma \left|\pi_{\mathbf{P}}(u) - \pi_{\mathbf{P}}(v)\right|_E^{-\alpha'} d\lambda(u)\, d\lambda(v)\, d\mathbf{P} \\
&\quad \leqslant \int_{B(R)^M} \int_\Sigma \int_\Sigma \left|\pi_{\mathbf{P}}(v)^{-1} * \pi_{\mathbf{P}}(u)\right|_E^{-\alpha'} d\lambda(u)\, d\lambda(v)\, d\mathbf{P} \\
&\quad \leqslant C(R, \mathbb{G}, \alpha') \int_\Sigma \int_\Sigma r_{u \wedge v}^{-\beta'} d\lambda(u)\, d\lambda(v)
\end{aligned}$$

$$
\begin{aligned}
&= C(R, \mathbb{G}, \alpha') \sum_{w \in W_*} \sum_{i \neq j} r_w^{-\beta'} \lambda(\Sigma_{wi}) \lambda(\Sigma_{wj}) \\
&\leqslant C(R, \mathbb{G}, \alpha') \sum_{w \in W_*} r_w^{\beta-\beta'} \lambda(\Sigma_w) \leqslant C(R, \mathbb{G}, \alpha') \sum_{m=1}^{\infty} 2^{-m(\beta-\beta')} \sum_{w \in W_m} \lambda(\Sigma_w).
\end{aligned}
$$

The latter expression is finite since $\sum_{w \in W_m} \lambda(\Sigma_w) = 1$ and $\beta > \beta'$. $\square$

In the proof of Lemma 5.4 we will make use of the following explicit representation for the symbolic representation map $\pi_{\mathbf{P}} : \Sigma \to K(\mathbf{P})$: if $u = u_1 u_2 \cdots u_m \cdots \in \Sigma$, then

$$
\begin{aligned}
\pi_{\mathbf{P}}(u) &= \lim_{m \to \infty} F_{u_1} \circ \cdots \circ F_{u_m}(o) \\
&= \lim_{m \to \infty} p_{u_1} * \delta_{r_{u_1}} \big(p_{u_2} * \delta_{r_{u_2}} \big(p_{u_3} * \cdots * \big(p_{u_m} * \delta_{r_{u_m}}(o) \big) \big) \big) \\
&= \lim_{m \to \infty} p_{u_1} * \delta_{r_{u_1}} p_{u_2} * \delta_{r_{u_1} r_{u_2}} p_{u_3} * \cdots * \delta_{r_{u_1} \cdots r_{u_{m-1}}} p_{u_m}.
\end{aligned}
\tag{5.8}
$$

Proof of Lemma 5.4. Let $0 < R < \infty$ and $0 \leqslant s \leqslant N$. By the Ball–Box Theorem, it suffices to verify (5.7) with the region of integration replaced by the M-fold product of CC balls $B_{cc}(R)^M$.

Using the notation $|x|_{cc} = d_{cc}(x, 0)$ let us observe that if $\mathbf{P} \in B_{cc}(0, R)$, then (5.8) and iterations of the triangle inequality for d_{cc} imply that

$$
\begin{aligned}
\big|\pi_{\mathbf{P}}(u)\big|_{cc} &\leqslant |p_{u_1}|_{cc} + r_{u_1} |p_{u_2}|_{cc} + r_{u_1} r_{u_2} |p_{u_3}|_{cc} + \cdots \\
&\leqslant R + \frac{1}{2} R + \frac{1}{4} R + \cdots = 2R,
\end{aligned}
\tag{5.9}
$$

hence

$$
\pi_{\mathbf{P}}(u) \in B_{cc}(0, 2R)
$$

for all $u \in \Sigma$ and $\mathbf{P} \in B_{cc}(R)$.

Now let $u, v \in \Sigma$, let $w = u \wedge v$, and assume that $|w| = k$. Since the maps F_j are CC similarities,

$$
d_{cc}\big(\pi_{\mathbf{P}}(u), \pi_{\mathbf{P}}(v)\big) = r_w d_{cc}\big(\pi_{\mathbf{P}}\big(\sigma^k u\big), \pi_{\mathbf{P}}\big(\sigma^k v\big)\big) \leqslant 4 R r_w
$$

by (5.9), so

$$
\pi_{\mathbf{P}}(v)^{-1} * \pi_{\mathbf{P}}(u) \in B_{cc}(0, 4 R r_w).
$$

By the Ball–Box Theorem, we conclude that

$$
\big[\pi_{\mathbf{P}}(v)^{-1} * \pi_{\mathbf{P}}(u)\big]_j \in \mathrm{Box}_E^{m_j}\big(\big(C' r_w\big)^j\big),
\tag{5.10}
$$

where C' depends only on R and C_{BB} and the Euclidean box in (5.10) is taken in $\mathbb{R}^{m_j}$.

Next, we record a useful explicit representation for the jth stratum of $\pi_{\mathbf{P}}(v)^{-1} * \pi_{\mathbf{P}}(u)$. Recall that $\mathbf{P} = (p_1, \ldots, p_M)$. We write $p_i \in \mathbb{G}$ in exponential coordinates: $p_i = (p_{i1}, \ldots, p_{is})$

where $p_{ij} \in \mathbb{R}^{m_j}$ for $j = 1, \dots, s$. Without loss of generality, we may assume that $u_{k+1} = 1$ and $v_{k+1} = 2$.

From (5.8) and (4.3) we find

$$\begin{aligned}
&\left[\pi_{\mathbf{P}}(v)^{-1} * \pi_{\mathbf{P}}(u)\right]_j \\
&\quad = r_w^j \left(p_{1j} - p_{2j} + \sum_{m=k+2}^{\infty} r_1^j r_{u_{k+2}}^j \cdots r_{u_{m-1}}^j p_{u_m,j} - r_2^j r_{v_{k+2}}^j \cdots r_{v_{m-1}}^j p_{v_m,j} \right) \\
&\qquad + \Theta_j(P_{1,j-1}, \dots, P_{M,j-1}) \\
&\quad = r_w^j \left(p_{1j} - p_{2j} + \sum_{i=1}^{M} E_{ij}(p_{ij}) \right) + \Theta_j(P_{1,j-1}, \dots, P_{M,j-1}),
\end{aligned} \tag{5.11}$$

where Θ_j is a real analytic function in the lower strata variables $P_{1,j-1}, \dots, P_{M,j-1}$ and $E_{1j}, \dots, E_{Mj} : \mathbb{R}^{m_j} \to \mathbb{R}^{m_j}$ are linear maps. In fact, each E_{ij} is just a standard Euclidean dilation of $\mathbb{R}^{m_j}$. Here we have used the notation from (4.2) for the cumulative lower strata variables P_{ij} associated to $p_i \in \mathbb{G}$. An explicit computation using (5.11) shows that

$$E_{ij}(x_j) = \rho_j x_j, \quad x_j \in \mathbb{R}^{m_j}, \tag{5.12}$$

where ρ_j is the sum over $m \in \mathbb{N}$ of terms of the form $\epsilon_{m,1} r_{\eta_{m,1}}^j + \epsilon_{m,2} r_{\eta_{m,2}}^j$ with $\eta_{m,1}, \eta_{m,2} \in W_m$ and $(\epsilon_{m,1}, \epsilon_{m,2}) \in \{(0,0), (+1,0), (0,-1), (+1,-1)\}$. Note that $\eta_{m,i}$ and $\epsilon_{m,i}$, $i = 1, 2$ (hence also ρ_j and E_{ij}) depend on u and v; see the middle expression in (5.11) for the explicit formula. For simplicity, we omit mention of this dependence in the notation.

The following argument is inspired by Falconer [20]. Our goal is to show that for each $0 \leqslant l \leqslant s-1$ the change of variables $\mathbf{P} \to \mathbf{P}$ defined by

$$\begin{aligned}
&p_{1j} \mapsto \begin{cases} q_j := [\pi_{\mathbf{P}}(v)^{-1} * \pi_{\mathbf{P}}(u)]_j, & j = 1, \dots, l+1, \\ p_{1j}, & j = l+2, \dots, s, \end{cases} \\
&p_{ij} \mapsto p_{ij}, \quad i = 2, \dots, M, \ j = 1, \dots, s,
\end{aligned} \tag{5.13}$$

is invertible.

Since $r_i \leqslant \gamma < \frac{1}{2}$ for all i and $j \geqslant 1$, (5.12) yields

$$\|E_{ij}\| = |\rho_j| \leqslant \sum_{m=1}^{\infty} \max\bigl\{|\epsilon_{m,1}|, |\epsilon_{m,2}|\bigr\} \gamma^{jm} \leqslant \sum_{m=1}^{\infty} \gamma^{jm} = \frac{\gamma^j}{1-\gamma^j} < 1$$

for each i, thus E_{ij} is a strict contraction and $I + E_{1j}$ is invertible with

$$\bigl\|(I + E_{1j})^{-1}\bigr\| \leqslant \frac{1-\gamma^j}{1-2\gamma^j}. \tag{5.14}$$

Using the lower triangular form of (5.11) it follows that the change of variables (5.13) is invertible. We compute its Jacobian determinant as:

$$dq_j = r_w^{jm_j} \det(I + E_{1j})\, dp_{1j} \tag{5.15}$$

and observe that

$$\det\big((I+E_{1j})^{-1}\big) \leqslant \big\|(I+E_{1j})^{-1}\big\|^{m_j} \leqslant \left(\frac{1-\gamma^j}{1-2\gamma^j}\right)^{m_j} \tag{5.16}$$

by Hadamard's inequality and (5.14).

Finally, we estimate

$$I := \int\limits_{B_{cc}(R)^M} \big|\pi_{\mathbf{P}}(v)^{-1} * \pi_{\mathbf{P}}(u)\big|_E^{-\alpha}\, d\mathbf{P}. \tag{5.17}$$

We fix $\ell = \ell(\alpha) \in \{0,\dots,s-1\}$ as in (2.5): l is the unique integer satisfying $\sum_{j=0}^{\ell} m_j < \alpha \leqslant \sum_{j=0}^{\ell+1} m_j$. We bound the integrand in (5.17) from above by

$$\big|\big(\big[\pi_{\mathbf{P}}(v)^{-1} * \pi_{\mathbf{P}}(u)\big]_1, \dots, \big[\pi_{\mathbf{P}}(v)^{-1} * \pi_{\mathbf{P}}(u)\big]_{\ell+1}\big)\big|_E^{-\alpha}.$$

Making the preceding change of variables and using (5.10), (5.15) and (5.16), we conclude that

$$I \leqslant C(R,\mathbb{G},\gamma) r_w^{-\sum_{j=0}^{\ell+1} j m_j} \times \int\limits_{\mathrm{Box}_E^{m_1}(C'r_w)} \cdots \int\limits_{\mathrm{Box}_E^{m_{\ell+1}}((C'r_w)^{l+1})} \big|(q_1,\dots,q_{\ell+1})\big|_E^{-\alpha}\, dq_{\ell+1}\cdots dq_1$$

or more simply,

$$I \leqslant C(R,\mathbb{G},\gamma) r_w^{-\sum_{j=0}^{\ell+1} j m_j} \int\limits_{\Pi_{\ell+1}\mathrm{Box}_{cc}(C'r_w)} |Q_{\ell+1}|_E^{-\alpha}\, dQ_{\ell+1}.$$

To conclude the proof, we write

$$\Pi_{\ell+1}\mathrm{Box}_{cc}(C'r_w) = \bigcup_{\sigma\subset S} A_\sigma,$$

where the union is taken over all nonempty subsets σ of $S = \{1,\dots,\ell+1\}$ and A_σ denotes the set of points $Q_{\ell+1} = (q_1,\dots,q_{\ell+1})$ in $\Pi_{\ell+1}\mathrm{Box}_{cc}(C'r_w)$ for which $|q_j| \leqslant (C'r_w)^{\ell+1}$ for all $j\in\sigma$. Then

$$\int\limits_{A_S} |Q_{\ell+1}|_E^{-\alpha}\, dQ_{\ell+1} \leqslant \int\limits_{B_E^{\sum_{j=0}^{\ell+1} m_j}(\sqrt{N}(C'r_w)^{\ell+1})} |Q_{\ell+1}|_E^{-\alpha}\, dQ_{\ell+1}$$

and

$$\int\limits_{A_\sigma} |Q_{\ell+1}|_E^{-\alpha}\, dQ_{\ell+1} \leqslant (C'r_w)^{\sum_{j\in S\setminus\sigma} j m_j} \int\limits_{\mathbb{R}^{\sum_{j\in\sigma} m_j}\setminus B_E^{\sum_{j\in\sigma} m_j}((C'r_w)^{\ell+1})} |Q_\sigma|_E^{-\alpha}\, dQ_\sigma$$

for $\sigma = \{\sigma_1, \dots, \sigma_{\#\sigma}\} \subsetneq S$ (with obvious notation $Q_\sigma = (q_{\sigma_1}, \dots, q_{\sigma_{\#\sigma}})$). In either case we obtain

$$\int_{A_\sigma} |Q_{\ell+1}|_E^{-\alpha}\, dQ_{\ell+1} \leqslant C(R, \mathbb{G}, \alpha) r_w^{(\ell+1)(\sum_{j=1}^{\ell+1} m_j - \alpha)};$$

summing over all nonempty subsets of S yields

$$I \leqslant C(R, \mathbb{G}, \alpha, \gamma) r_w^{-\sum_{j=0}^{\ell+1} j m_j} r_w^{(\ell+1)(\sum_{j=0}^{\ell+1} m_j - \alpha)} = C(R, \mathbb{G}, \alpha, \gamma) r_w^{-\beta_-(\alpha)}.$$

This completes the proof of Lemma 5.4. □

6. Jet spaces

Our goal in this section is twofold. First, we illustrate the main results of this paper in a well-known explicit class of Carnot groups: the jet spaces $J^k(\mathbb{R}, \mathbb{R})$. In the standard Carnot group presentation of $J^k(\mathbb{R}, \mathbb{R})$, similarity maps are polynomial in the underlying Euclidean geometry.

In the second part of the section we describe an alternate Carnot group presentation of $J^k(\mathbb{R}, \mathbb{R})$ in which similarities are **affine** maps in the Euclidean geometry and the constituent linear maps are given by triangular matrices. We then relate our work to that of Falconer and Miao in [18].

6.1. Jet spaces as Carnot groups: the classical model

References for this material include Section 6.4 in [50, §6.4], [61] and [62]. General discussions of the geometry of jet spaces and jet bundles can be found in [54] and [56].

The kth order Taylor polynomial of a C^k function $\mathbf{f}: \mathbb{R} \to \mathbb{R}$ at a point $x_0 \in \mathbb{R}$ is

$$\big(T_{x_0}^k \mathbf{f}\big)(\xi) = \sum_{i=0}^{k} \mathbf{f}^{(i)}(x_0) \frac{(\xi - x_0)^i}{i!}.$$

Two functions $\mathbf{f}_1, \mathbf{f}_2 \in C^k(\mathbb{R})$ are defined to be *equivalent* at x_0, written $\mathbf{f}_1 \sim_{x_0} \mathbf{f}_2$, if $T_{x_0}^k \mathbf{f}_1 = T_{x_0}^k \mathbf{f}_2$. The equivalence class of $\mathbf{f}$ is the *k-jet of* $\mathbf{f}$ *at* x_0, denoted $\mathrm{jet}_{x_0}^k(\mathbf{f})$. The *kth order jet space* is

$$J^k(\mathbb{R}, \mathbb{R}) := \bigcup_{x_0 \in \mathbb{R}} C^k(\mathbb{R})/\sim_{x_0}.$$

We identify $J^k(\mathbb{R}, \mathbb{R})$ with the Euclidean space $\mathbb{R}^{k+2}$ by introducing coordinates $x : J^k(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$ and $u_j : J^k(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$, $0 \leqslant j \leqslant k$, where $x(\mathrm{jet}_{x_0}^k(\mathbf{f})) = x_0$ and $u_j(\mathrm{jet}_{x_0}^k(\mathbf{f})) = \mathbf{f}^{(j)}(x_0)$. In this coordinate system we will write elements of $J^k(\mathbb{R}, \mathbb{R})$ as $(k+2)$-tuples

$$p = \big(x, u^{(k)}\big) = (x, u_k, \dots, u_0).$$

Contact and horizontal structures in $J^k(\mathbb{R}, \mathbb{R})$. The k-jet of a map $\mathbf{f} \in C^k(\mathbb{R})$ is the section $x_0 \mapsto \mathrm{jet}_{x_0}^k(\mathbf{f})$ of the bundle $x : J^k(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$. A contact form θ on $J^k(\mathbb{R}, \mathbb{R})$ is a 1-form satisfying $(\mathrm{jet}^k_\bullet(\mathbf{f}))^*\theta = 0$ for all k-jets $\mathrm{jet}^k_\bullet(\mathbf{f})$. By the chain rule, the cotangent space is framed by the collection of 1-forms dx, $\omega_k = du_k$, and $\omega_j = du_j - u_{j+1}dx$ where $j = 0, \dots, k-1$.

The horizontal tangent bundle $\mathcal{H}$ is defined pointwise by

$$\mathcal{H}_p = \bigl\{ V \in T_p J^k(\mathbb{R}, \mathbb{R})\colon\ \omega_j(V) = 0 \text{ for all } j = 0, \dots, k-1 \bigr\}.$$

In coordinates, $V = dx(V)X + \omega_k(V)U_k$, where $X = \frac{\partial}{\partial x} + u_k \frac{\partial}{\partial u_{k-1}} + \cdots + u_1 \frac{\partial}{\partial u_0}$ and $U_j = \frac{\partial}{\partial u_j}$ for $j = 0, \dots, k$. We note the nontrivial commutation relations

$$[U_j, X] = U_{j-1}, \quad j = 1, \dots, k. \tag{6.1}$$

Setting $V_1 = \mathcal{H} = \mathrm{span}\{X, U_k\}$ and $V_j = \mathrm{span}\{U_{k-j+1}\} = [V_1, V_{j-1}]$ for $j = 2, \dots, k+1$, we obtain a $(k+1)$-step nilpotent Lie algebra $\mathfrak{j}^k = \mathfrak{j}^k(\mathbb{R}, \mathbb{R}) = \mathfrak{v}_1 \oplus \cdots \oplus \mathfrak{v}_{k+1}$ which gives $J^k(\mathbb{R}, \mathbb{R})$ the structure of a $(k+1)$-step Carnot group.

The homogeneous dimension of $J^k(\mathbb{R}, \mathbb{R})$ is $Q = 1 + \binom{k+2}{2}$ while the underlying Euclidean space is $\mathbb{R}^{k+2}$. The bases $\{X, U_k, \dots, U_0\}$ and $\{dx, \omega_k, \dots, \omega_0\}$ are dual. Note that it is the vector fields X and U_k which define the horizontal directions in this presentation.

Remark 6.1. $\mathfrak{j}^1$ is isomorphic to the Lie algebra of the first Heisenberg group, and $\mathfrak{j}^2$ is isomorphic to the Lie algebra of the Engel group. In general, $\mathfrak{j}^k$ is known as the kth *Goursat algebra* or *model filiform algebra*. It arises naturally in control theory as the configuration space for optimal path planning in the kinematics of multi-stage trailers, cf. [48].

The group law, dilations and similarities in $J^k(\mathbb{R}, \mathbb{R})$. Using the above introduced, so-called second kind coordinates, the group law reads as follows:

$$\bigl(x, u^{(k)}\bigr) \odot \bigl(y, v^{(k)}\bigr) = \bigl(z, w^{(k)}\bigr),$$

where $z = x + y$ and

$$w_j = v_j + \sum_{l=j}^{k} u_l \frac{y^{l-j}}{(l-j)!}, \quad 0 \leqslant j \leqslant k. \tag{6.2}$$

The dilation of $J^k(\mathbb{R}, \mathbb{R})$ by scaling factor r is

$$\delta_r\bigl(x, u^{(k)}\bigr) = \bigl(rx, ru_k, r^2 u_{k-1}, \dots, r^{k+1} u_0\bigr). \tag{6.3}$$

From (6.2) it follows that similarities in $J^k(\mathbb{R}, \mathbb{R})$ are given by polynomials of degree $k+1$ in this model.

In the setting of $J^k(\mathbb{R}, \mathbb{R})$, Theorems 2.4 and 2.6 read as follows.

Theorem 6.2. *For $S \subset J^k(\mathbb{R}, \mathbb{R})$, $\beta_-(\dim_E(S)) \leqslant \dim_{cc}(S) \leqslant \beta_+(\dim_E(S))$, where the upper dimension comparison function for $J^k(\mathbb{R}, \mathbb{R})$ is*

$$\beta_+(\alpha) = \begin{cases} (k-l+1)\alpha + \binom{l+1}{2}, & \alpha \in [l, l+1],\ l = 0, \dots, k-1, \\ \alpha + \binom{k+1}{2}, & \alpha \in [k, k+2], \end{cases} \tag{6.4}$$

and the lower dimension comparison function for $J^k(\mathbb{R},\mathbb{R})$ is

$$\beta_-(\alpha)=\begin{cases}\alpha, & \alpha\in[0,2],\\ (l+1)\alpha+1-\binom{l+2}{2}, & \alpha\in[l+1,l+2],\ l=1,\dots,k.\end{cases}\tag{6.5}$$

6.2. Jet spaces as Carnot groups: an alternate model

We now describe another Carnot group model for the jet space $J^k(\mathbb{R},\mathbb{R})$. The principal advantage of this model, in the context of this paper, is that left translation is given by affine maps in the underlying Euclidean geometry. Thus CC self-similar IFSs are Euclidean self-affine. This gives us the possibility to compare our results with the recent work of Falconer and Miao in [18].

In this model, we identify $J^k(\mathbb{R},\mathbb{R})$ with $\mathbb{R}^{k+2}$ by introducing a different set of coordinates: $x: J^k(\mathbb{R},\mathbb{R})\to\mathbb{R}$ and $\tilde u_j: J^k(\mathbb{R},\mathbb{R})\to\mathbb{R}$, $0\leqslant j\leqslant k$, where $x(\mathrm{jet}^k_{x_0}(\mathbf{f}))=x_0$ and

$$\tilde u_j\big(\mathrm{jet}^k_{x_0}(\mathbf{f})\big)=\frac{\partial}{\partial\xi^j}\big(T^k_{x_0}\mathbf{f}\big)(\xi)\big|_{\xi=0}=\sum_{i=j}^k f^{(i)}(x_0)\frac{(-x_0)^{i-j}}{(i-j)!}.\tag{6.6}$$

In these coordinates we will write elements of $J^k(\mathbb{R},\mathbb{R})$ as $(k+2)$-tuples $p=(x,\tilde u^{(k)})=(x,\tilde u_k,\dots,\tilde u_0)$. We obtain from (6.6) the coordinate transformation $\phi(x,u^{(k)})=(x,\tilde u^{(k)})$, where

$$\tilde u_j=\sum_{i=j}^k u_i\frac{(-x)^i}{i!},$$

which converts between the two models. Indeed, we define the group law so that ϕ becomes an isomorphism, setting

$$\big(x,\tilde u^{(k)}\big)*\big(y,\tilde v^{(k)}\big)=\phi\big(\phi^{-1}\big(x,\tilde u^{(k)}\big)\odot\phi^{-1}\big(y,\tilde v^{(k)}\big)\big)=\big(x+y,\tilde w^{(k)}\big)$$

and

$$\tilde w_j=\tilde u_j+\sum_{l=j}^k\tilde v_l\frac{(-x)^{l-j}}{(l-j)!},\quad 0\leqslant j\leqslant k.$$

It now follows that the left invariant vector fields are given by

$$\tilde X=\frac{\partial}{\partial x}\quad\text{and}\quad\tilde U_j=\frac{\partial}{\partial\tilde u_j}-x\frac{\partial}{\partial\tilde u_{j-1}}+\cdots+\frac{1}{j!}(-x)^j\frac{\partial}{\partial\tilde u_0},$$

where $0\leqslant j\leqslant k$, and we observe that these vector fields satisfy the nontrivial commutation relations

$$[\tilde U_j,\tilde X]=\tilde U_{j-1},\quad j=1,\dots,k.\tag{6.7}$$

Thus $\mathfrak{j}^k(\mathbb{R},\mathbb{R}) = \mathfrak{v}_1 \oplus \cdots \oplus \mathfrak{v}_{k+1}$ where $\mathfrak{v}_1, \ldots, \mathfrak{v}_{k+1}$ correspond to the vector bundles $V_1 = \operatorname{span}\{\tilde{X}, \tilde{U}_k\}$ and $V_j = \operatorname{span}\{\tilde{U}_{k-j+1}\}$ for $j = 2, \ldots, k$. The dual forms are $dx, d\tilde{u}_k, \tilde{\omega}_{k-1}, \ldots, \tilde{\omega}_0$ where

$$\tilde{\omega}_j = \sum_{\ell=j}^{k} \frac{x^{\ell-j}}{(\ell-j)!}\, d\tilde{u}_\ell$$

for $j = 0, \ldots, k-1$. In this model the dilation by scaling factor r is

$$\delta_r\big(x, \tilde{u}^{(k)}\big) = \big(rx, r\tilde{u}_k, r^2\tilde{u}_{k-1}, \ldots, r^{k+1}\tilde{u}_0\big). \tag{6.8}$$

We emphasize again the crucial feature of this model: left translation $(y, \tilde{v}^{(k)}) \to (x, \tilde{u}^{(k)}) * (y, \tilde{v}^{(k)})$ is an affine map of the underlying Euclidean space $\mathbb{R}^{k+2}$. With dilations δ_r defined as in (6.8), we see that the CC similarity

$$\big(z, \tilde{w}^{(k)}\big) = F\big(x, \tilde{u}^{(k)}\big) = \big(a, \tilde{b}^{(k)}\big) * \delta_r\big(x, \tilde{u}^{(k)}\big),$$

for fixed $p_0 = (a, \tilde{b}^{(k)}) \in J^k(\mathbb{R},\mathbb{R})$, takes the form $z = rx + a$ and

$$\tilde{w}_j = \sum_{l=j}^{k} r^{k+1-l}\tilde{u}_l \frac{(-a)^{l-j}}{(l-j)!} + \tilde{b}_j$$

for $0 \leqslant j \leqslant k$. Observe that F is a Euclidean affine map of the form

$$\begin{pmatrix} z \\ \tilde{w}_k \\ \vdots \\ \tilde{w}_1 \\ \tilde{w}_0 \end{pmatrix} = \begin{pmatrix} r & & & & & \\ 0 & r & & & & \\ 0 & -ra & r^2 & & & \\ \vdots & \vdots & \ddots & \ddots & & \\ 0 & \frac{r(-a)^{k-1}}{(k-1)!} & \cdots & -r^{k-1}a & r^k & \\ 0 & \frac{r(-a)^k}{k!} & \cdots & \frac{r^{k-1}a^2}{2} & -r^k a & r^{k+1} \end{pmatrix} \begin{pmatrix} x \\ \tilde{u}_k \\ \vdots \\ \tilde{u}_1 \\ \tilde{u}_0 \end{pmatrix} + \begin{pmatrix} a \\ \tilde{b}_k \\ \vdots \\ \tilde{b}_1 \\ \tilde{b}_0 \end{pmatrix}. \tag{6.9}$$

6.3. A relation between self-similar sub-Riemannian fractal geometry and self-affine Euclidean fractal geometry in jet spaces

Let us recall that Theorem 5.1 of Falconer gives an explicit expression for the almost sure dimension of the invariant sets of Euclidean self-affine iterated function systems which involves taking a limit of an average of the singular value functions of iterated products of the constituent linear maps $A_1, \ldots, A_M$. Formula (5.2) is in many cases difficult to use in practice due to the presence of the limit, and further work has been done to identify specific situations where the calculation can be streamlined. In [18], Falconer and Miao provide a simple closed-form expression for the critical exponent $d(\mathbf{A})$ in case the matrices A_i are upper triangular. According to Corollary 2.6 in [18], for a collection $\mathbf{A}$ of contractive upper triangular matrices $A_1, \ldots, A_M$ on $\mathbb{R}^n$, the critical exponent $d(\mathbf{A})$ defined in (5.2) can be recovered as follows: Let $a^i_{jj'}$ denote the (j, j')th entry in the matrix A_i. For $0 < t \leqslant n$ define $D(t)$ piecewise on the subintervals $m-1 < t \leqslant m$, $m \in \{1, \ldots, n\}$, as follows: for $m = 1$ set

$$D(t) = \max_{j_1'} \sum_{i=1}^{M} \left|a^i_{j_1' j_1'}\right|^t, \tag{6.10}$$

and for $2 \leqslant m$ set

$$D(t) = \max_{\substack{\{j_1,\ldots,j_{m-1}\}\\ \{j_1',\ldots,j_m'\}}} \sum_{i=1}^{M} \left|\prod_{\ell=1}^{m-1} a^i_{j_\ell j_\ell}\right|^{m-t} \left|\prod_{\ell=1}^{m} a^i_{j_\ell' j_\ell'}\right|^{t-m+1}, \tag{6.11}$$

where the maximum is taken over all $(m-1)$-tuples $\{j_1, \ldots, j_{m-1}\}$ and m-tuples $\{j_1', \ldots, j_m'\}$ with distinct entries in $\{1, \ldots, n\}$. The critical exponent $d(\mathbf{A})$ is the unique t such that $D(t) = 1$.

It is worth emphasizing the fact that the above expression for the critical exponent depends only on the diagonal entries of the matrices A_i. This will be important for us later on in this section.

The fact that the matrix part of F is lower triangular is a feature of our presentation of $J^k(\mathbb{R}, \mathbb{R})$; this minor discrepancy with the Falconer–Miao formalism is immaterial. One can either permute the coordinates in $J^k(\mathbb{R}, \mathbb{R})$ so that the matrix part becomes upper triangular, or restate the results of [18] for lower triangular matrices.

The following statement connects Theorem 2.8 to the result of Falconer–Miao.

Proposition 6.3. *Fix $r_1, \ldots, r_M < 1$ so that $\sum_{i=1}^M r_i^\beta = 1$ and let $A_1 = A_1(r_1, a_1), \ldots, A_M = A_M(r_M, a_m)$ be matrices in the form* (6.9). *Then the equality*

$$\beta = \beta_-\bigl(d(\mathbf{A})\bigr) \tag{6.12}$$

holds, where $d(\mathbf{A})$ is the Falconer–Miao critical exponent defined above by the relations (6.10), (6.11) *and the condition $D(d(\mathbf{A})) = 1$.*

Proof. Fix $r_1, \ldots, r_M < 1$ so that $\sum_{i=1}^M r_i^\beta = 1$ and let $A_1 = A_1(r_1, a_1), \ldots, A_M = A_M(r_M, a_m)$ be matrices in the form which occur in (6.9). We observe that the jth diagonal entry of $A_i(r_i, a_i)$ is

$$r_i^{\max\{j-1,1\}},$$

where $1 \leqslant j \leqslant k+2$.

We consider the expressions in (6.10) and (6.11). For $m = 1$, the maximum in (6.10) occurs when $j_1' = 1$ and we have

$$D(t) = \sum_{i=1}^{M} r_i^t = \sum_{i=1}^{M} r_i^{\beta_-(t)},$$

where $0 < t \leqslant 1$. For $2 \leqslant m$, the maximum in (6.11) occurs when $j_\ell = j_\ell' = \ell$. Furthermore the parameter l from Theorem 2.4 is given by $l = m-2$, and

$$D(t) = \sum_{i=1}^{M} r_i^{(1+1+2+\cdots+(m-2))(m-t)+(1+1+2+\cdots+(m-1))(t-m+1)}$$
$$= \sum_{i=1}^{M} r_i^{(m-1)t+1-\binom{m}{2}} = \sum_{i=1}^{M} r_i^{(l+1)t+1-\binom{l+2}{2}} = \sum_{i=1}^{M} r_i^{\beta_-(t)},$$

where $m-1 < t \leqslant m$. (See (6.5).) Thus $D(t) = \sum_{i=1}^{M} r_i^{\beta_-(t)}$ for all $t \in [0, k+2]$ and we conclude that $\beta_-(d(\mathbf{A}))$ coincides with the similarity dimension of any CC self-similar IFS in $J^k(\mathbb{R}, \mathbb{R})$ whose matrix parts are the given matrices $A_1, \ldots, A_M$. $\square$

As a corollary to Theorem 2.8, we observe that the dimension of the invariant set in $\mathbb{R}^{k+2}$ for a self-affine IFS consisting of maps of the form (6.9) is equal to $d(\mathbf{A})$ almost surely. We must point out an important caveat. Falconer and Miao treat the case when the linear parts are fixed upper triangular matrices and the translation parameters vary. However, as is emphasized in [18], the expression for $D(t)$ in (6.10) and (6.11) depends only on the diagonal entries of the matrices A_i. It is therefore reasonable to expect that the value of $d(\mathbf{A})$ continues to provide the correct almost sure dimension even if variation is allowed in the linear parts, provided it only occurs in off-diagonal entries. In (6.9), we see that the matrices which arise in CC similarities of $J^k(\mathbb{R}, \mathbb{R})$ have precisely this dependence on the translation parameters and this expectation is confirmed by Theorem 2.8.

Can one prove the Falconer–Miao almost sure dimension formula in a more general situation, when the off-diagonal entries depend on the parameters in a more general manner than (6.9)?

Remark 6.4. The general jet space Carnot groups $J^k(\mathbb{R}^m, \mathbb{R}^n)$ (see [62]) also admit a presentation in which left translation is a Euclidean affine map. Analogs of the above results continue to hold in this setting. It would be interesting to characterize the class of Carnot groups which admit a presentation in which left translations are affine maps in the underlying Euclidean geometry, and to relate Theorem (2.8) to the results of Falconer–Miao in that case.

Remark 6.5. We conclude this section by describing the solution to Gromov's problem 1.1 in the Engel group $\mathbb{E} = J^2(\mathbb{R}, \mathbb{R})$. Thus we compute $\beta_k := \inf\{\dim_{cc}^H S\colon\ S \subset \mathbb{E} \text{ compact},\ \dim_{top} S = k\}$ for each $k = 0, 1, 2, 3, 4$. Recall that $\mathbb{E}$ is a step three Carnot group with topological dimension $N = \dim_{top} \mathbb{E} = 4$ and CC Hausdorff dimension $Q = \dim_{cc}^H \mathbb{E} = 7$. The values $\beta_0 = 0$, $\beta_1 = 1$, and $\beta_4 = 7$ are obvious. According to [30, §2.1], we have $\beta_3 = 6$. We claim that

$$\beta_2 = 3. \tag{6.13}$$

We note that the projection $\Pi_2 : \mathbb{E} = J^2(\mathbb{R}, \mathbb{R}) \to J^1(\mathbb{R}, \mathbb{R}) = \mathbb{H}^1$ is 1-Lipschitz when domain and target are equipped with their CC metric. Suppose that $S \subset \mathbb{E}$ has topological dimension two. If $\Pi_2(S)$ also has topological dimension two, then $\dim_{cc}^H \Pi_2(S) \geqslant 3$ by (1.3) and hence $\dim_{cc}^H S \geqslant 3$. On the other hand, if $\Pi_2(S)$ has topological dimension one, then

$$\dim_{top} (\Pi_2)^{-1}(p) \cap S \geqslant 1$$

for at least one point $p \in \Pi_2(S)$ [36, Theorem VI.7]. In particular,

$$\dim_E^H (\Pi_2)^{-1}(p) \cap S \geqslant 1.$$

The CC metric on $\mathbb{E}$ restricted to the fiber $(\Pi_2)^{-1}(p)$ is a multiple of $d_E^{1/3}$. Hence

$$\dim_{cc}^H S \geqslant \dim_{cc}^H (\Pi_2)^{-1}(p) \cap S \geqslant 3.$$

In all cases, we conclude that $\dim_{cc}^H S \geqslant 3$. The proof of (6.13) is complete.

7. Another example

To further illustrate the principal application of our theory to the computation of dimensions of nonlinear Euclidean fractals, we describe another example of a three-dimensional horizontal fractal in a six-dimensional Carnot group of step four.

We consider the three-step nilpotent Lie algebra $\mathfrak{g}$ modeled by strictly upper triangular matrices of the form

$$A = \begin{pmatrix} 0 & x_1 & x_3 & x_4 & x_6 \\ 0 & 0 & x_2 & -x_3 & x_5 \\ 0 & 0 & 0 & x_1 & x_3 \\ 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and let $\mathbb{G}$ be the associated nilpotent Lie group. We identify $\mathfrak{g}$ with $\mathbb{R}^6$ via the correspondence $A \leftrightarrow (x_1, x_2, x_3, x_4, x_5, x_6)$. We denote by e_i the ith standard basis element in $\mathbb{R}^6$, and will use the same notation to refer to the corresponding element of $\mathfrak{g}$. This Lie algebra admits a stratified vector space decomposition $\mathfrak{g} = \mathfrak{v}_1 \oplus \mathfrak{v}_2 \oplus \mathfrak{v}_3 \oplus \mathfrak{v}_4$, where $\mathfrak{v}_1 = \operatorname{span}\{e_1, e_2\}$, $\mathfrak{v}_2 = \operatorname{span}\{e_3\}$, $\mathfrak{v}_3 = \operatorname{span}\{e_4, e_5\}$ and $\mathfrak{v}_4 = \operatorname{span}\{e_6\}$. We observe the relations $[e_1, e_2] = e_3$, $[e_1, e_3] = -2e_4$, $[e_2, e_3] = 2e_5$, $[e_1, e_5] = [e_4, e_2] = e_6$, all other brackets being equal to zero. Upon introducing an inner product on $\mathfrak{g}$ so that the subspaces $\mathfrak{v}_i$ are orthogonal, we equip $\mathbb{G}$ with the structure of a four-step Carnot group of dimension $N = 6$ with strata dimensions $m_1 = 2$, $m_2 = 1$, $m_3 = 2$ and $m_4 = 1$. The homogeneous dimension is $Q = 14$. The upper and lower dimension comparison functions for this group are easily computed to be

$$\beta_+(\alpha) = \min\{4\alpha, 3\alpha + 1, 2\alpha + 4, \alpha + 8\}$$

and

$$\beta_-(\alpha) = \max\{\alpha, 2\alpha - 2, 3\alpha - 5, 4\alpha - 10\}.$$

The Carnot group multiplication is given in second kind coordinates by $x \odot y = z$, where $z_1 = x_1 + y_1$, $z_2 = x_2 + y_2$, $z_3 = x_3 + y_3 - x_2 y_1$, $z_4 = x_4 + y_4 + 2x_3 y_1 - x_2 y_1^2$, $z_5 = x_5 + y_5 + 2x_2 y_1 y_2 - 2x_3 y_2 + x_2^2 y_1$, and $z_6 = x_6 + y_6 - x_5 y_1 + x_4 y_2 - x_2 y_1^2 y_2 + 2x_3 y_1 y_2 - \frac{1}{2} x_2^2 y_1^2$. Observe that left translation is given by **cubic** maps in the underlying Euclidean geometry. Dilations in this group are of the form $\delta_r(x) = (rx_1, rx_2, r^2 x_3, r^3 x_4, r^3 x_5, r^4 x_6)$. The projection $\Pi_3 : \mathbb{R}^6 \to \mathbb{R}^3$ given by $x \mapsto (x_1, x_2, x_3)$ functions as a sub-Riemannian projection (in particular, as a contractive map) from $\mathbb{G} = \mathbb{R}^6$ to the first jet space $J^1(\mathbb{R}, \mathbb{R}) = \mathbb{R}^3$.

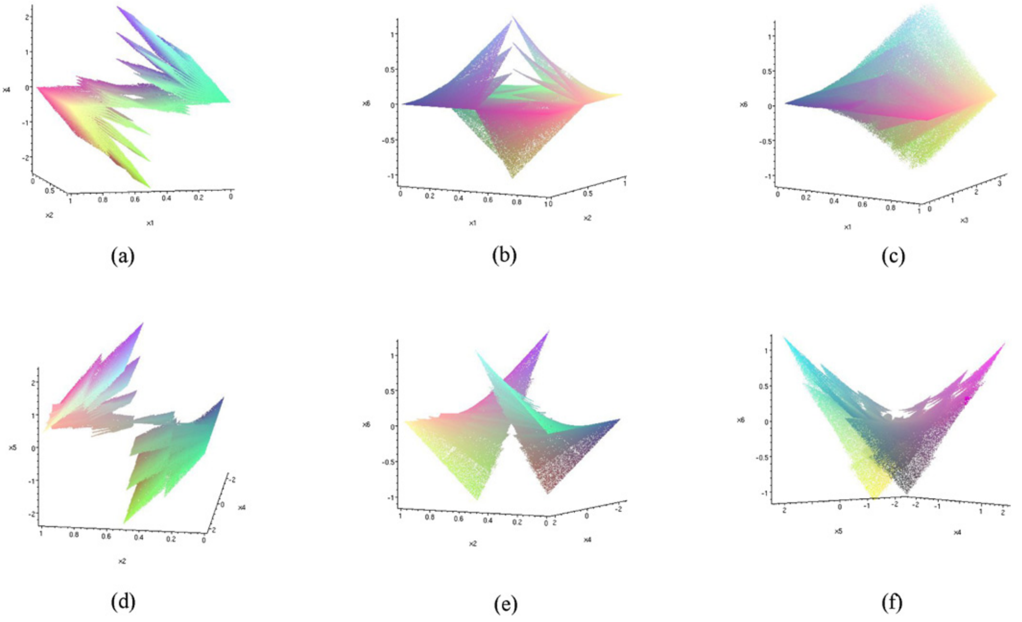


Fig. 5. Projections of the set S from Proposition 7.1: (a) into $x_1x_2x_4$-space, (b) into $x_1x_2x_6$-space, (c) into $x_1x_3x_6$-space, (d) into $x_2x_4x_5$-space, (e) into $x_2x_4x_6$-space, (f) into $x_4x_5x_6$-space.

Proposition 7.1. *Let $\{F_i\}_{1\leqslant i\leqslant 16}$ be the Carnot–Carathéodory self-similar iterated function system in $\mathbb{G}$ consisting of the maps $F_i(x) = p_i \odot \delta_{1/2}(p_i^{-1} \odot x)$, where the points p_i enumerate the set*

$$\bigl\{(i, j, k, 0, 0, 0)\colon\ i \in \{0, 1\},\ j \in \{0, 1\},\ k \in \{0, 1, 2, 3\}\bigr\},$$

and let S be the invariant set for this IFS. Then $\dim_{cc} S = 4$ and $\dim_E S = 3$.

Proof. The projection $\Pi_3(S)$ of this IFS into $J^1(\mathbb{R}, \mathbb{R})$ coincides with the IFS defining the Strichartz tile $T_2 \subset \mathbb{H}^1$, under the identification of $J^1(\mathbb{R}, \mathbb{R})$ with $\mathbb{H}^1$ discussed in Section 6. Thus $\dim_{cc} S \geqslant 4$ and $\dim_E S \geqslant 3$ by (2.12) and (2.13). On the other hand, since 4 is the similarity dimension of the defining IFS $\{F_i\}$, we also have $\dim_{cc} S \leqslant 4$ by Theorem 4.11. Hence $\dim_{cc} S = 4$, and then also $\dim_E S = 3$ by Theorem 2.4. □

Fig. 5 shows projections of S into various three-dimensional subspaces of $\mathbb{R}^6$. Curiously, these pictures suggest that all of these coordinate projections have dimension strictly less than three. Generic three-dimensional projections of a set $S \subset \mathbb{R}^6$ of Hausdorff dimension three again have Hausdorff dimension three, see, e.g., Corollary 9.4 in [47].

8. Open problems and questions

We conclude with remarks, problems and questions motivated by these investigations.

8.1. Remarks concerning Problems 1.1 and 1.2

Recall that Problems 1.1 and 1.2 ask for characterizations of

$$\beta_k = \inf\bigl\{\dim_{cc}^H S\colon\ S \subset M \text{ compact},\ \dim_{top} S = k\bigr\}$$

and

$$\Delta(M) = \bigl\{(k,\alpha,\beta)\colon\ \exists S \subset M,\ \dim_{top} S = k,\ \dim_d^H S = \alpha,\ \dim_{d_0}^H S = \beta\bigr\}$$

respectively. For each $k = 0, 1, \dots, \dim M$, define

$$\Delta_k(M) = \bigl\{(\alpha,\beta)\colon\ \exists S \subset M,\ \dim_{top} S = k,\ \dim_d^H S = \alpha,\ \dim_{d_0}^H S = \beta\bigr\}.$$

Thus $\Delta(M) = \bigcup_{k=0}^{\dim M} \{k\} \times \Delta_k(M)$ and $\Delta'(M) = \bigcup_{k=0}^{\dim M} \Delta_k(M)$. In case $M = \mathbb{G}$ is a Carnot group, Theorems 2.4 and 2.6 yield

$$\Delta_0(\mathbb{G}) = \Delta'(\mathbb{G}) = \bigl\{(\alpha,\beta)\colon\ 0 \leqslant \alpha \leqslant \dim M,\ \beta_-(\alpha) \leqslant \beta \leqslant \beta_+(\alpha)\bigr\}.$$

We conjecture that the infimum in the definition of β_k is realized on a smooth k-dimensional submanifold. More precisely, we make the following

Conjecture 8.1. *For each k, let $\mathfrak{w}$ be a k-dimensional Lie subalgebra of $\mathfrak{g}$ so that the quantity*

$$d(\mathfrak{w}) = \sum_{j=1}^{s} j \dim\bigl(\mathfrak{w} \cap (\mathfrak{v}_1 + \cdots + \mathfrak{v}_j)/\mathfrak{w} \cap (\mathfrak{v}_1 + \cdots + \mathfrak{v}_{j-1})\bigr)$$

is minimized among all k-dimensional Lie subalgebras of $\mathfrak{g}$. Then $\beta_k = d(\mathfrak{w}) = \dim_{cc}^H S$ for any compact set $S \subset \exp(\mathfrak{w})$ with nonempty relative interior.

Let us point out some consequences of a positive answer to Conjecture 8.1.

Remarks 8.2. (1) If Conjecture 8.1 is true, then $\{\dim_{cc}^H S\colon\ S \subset \mathbb{G} \text{ Borel},\ \dim_{top} S = k\} = [\beta_k, \dim_{cc}^H \mathbb{G}]$ for each k.

It suffices to verify that every $\beta \in [\beta_k, \dim_{cc}^H \mathbb{G}]$ arises as the CC Hausdorff dimension of a Borel set S of topological dimension k. Using Theorem 2.6, we find a bounded Borel set S_0 of topological dimension zero with $\dim_E^H S_0 = k$ and $\dim_{cc}^H S_0 = \beta$. Let S_1 be a compact set of topological and Euclidean Hausdorff dimension k with $\dim_{cc}^H S_1 = \beta_k$ (see the statement of Conjecture 8.1). Then $S = S_0 \cup S_1$ is a bounded Borel set of topological dimension k (see Theorem III.2 in [36]) whose CC Hausdorff dimension is β.

Note also that regardless of the validity of Conjecture 8.1, the preceding argument shows that the set $\{\dim_{cc}^H S\colon\ S \subset \mathbb{G} \text{ Borel},\ \dim_{top} S = k\}$ is an interval, i.e., either $(\beta_k, \dim_{cc}^H \mathbb{G}]$ or $[\beta_k, \dim_{cc}^H \mathbb{G}]$.

(2) If Conjecture 8.1 is true, then $\Delta_k(M) = \Delta'(M) \cap [k, \dim M] \times [\beta_k, \dim_{cc}^H M]$ for each k.

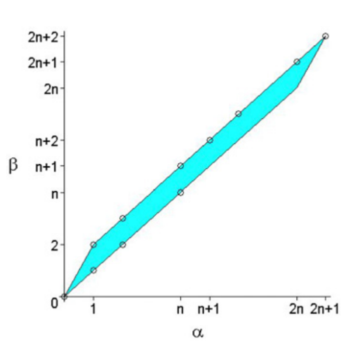


Fig. 6. Solution to Problems 1.3 and 8.3 in $\mathbb{H}^n$.

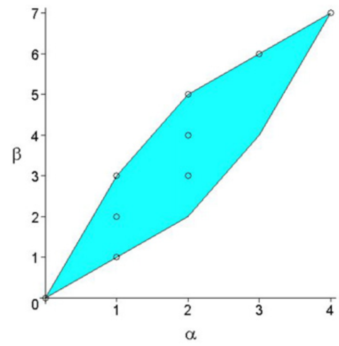


Fig. 7. Solution to Problems 1.3 and 8.3 in $\mathbb{E}$.

The argument is quite similar. Again, it suffices to verify that every pair $(\alpha, \beta) \in [k, \dim \mathbb{G}] \times [\beta_k, \dim_{cc}^H \mathbb{G}]$ with $\beta_-(\alpha) \leqslant \beta \leqslant \beta_+(\alpha)$ arises as the dimension pair for a Borel set $S \subset \mathbb{G}$ of topological dimension k. Theorem 2.6 yields a bounded Borel set S_0 of topological dimension zero with $\dim_E^H S_0 = \alpha$ and $\dim_{cc}^H S_0 = \beta$. Let S_1 be a compact set of topological and Euclidean Hausdorff dimension k and CC Hausdorff dimension β_k. Then $S = S_0 \cup S_1$ is a bounded Borel set with the desired properties.

8.2. *Dimension comparison for submanifolds*

Gromov's dimension comparison problem could be stated for various classes of sets. While our results offer the full solution for general sets, it would be interesting to study the problem for a more restricted class of smooth manifolds. To state formally the problem let us use notation from the introduction and denote by $\mathcal{S}(M)$ the class of smooth submanifolds of M. We can now formulate the question as follows:

Problem 8.3. Determine exactly the set

$$\Delta'_{\mathcal{S}}(M) := \bigl\{(\alpha, \beta) \in \mathbb{R}^2\colon\ (\alpha, \beta)(N) = \bigl(\dim N, \dim_{d_0}^H N\bigr),\ N \in \mathcal{S}(M)\bigr\}. \tag{8.1}$$

The solution (1.4) to this problem in $\mathbb{H}^1$ hints at the inherent difficulties, which exceed those involved in our solution to dimension comparison problem for general sets. Indeed, while the solution to Problem 1.3 involves only the strata dimensions, the solution to Problem 8.3 involves the structure of the Lie algebra, specifically, the commutation relations. We indicate in Figs. 6 and 7 the solution to Problem 8.3 in the Heisenberg groups $\mathbb{H}^n$ and the Engel group $\mathbb{E}$, superimposed on the regions $\Delta'(\mathbb{H}^n)$ and $\Delta'(\mathbb{E})$. Note that certain points with integral coordinates in $\Delta'(\mathbb{E})$ are omitted in Fig. 7. In fact, by Remark 6.5, $\mathbb{E}$ contains no surfaces with CC dimension 2, nor any 3-dimensional hypersurfaces with CC dimension 4 or 5.

For further illustration, we list the CC Hausdorff dimensions of the coordinate subspaces of $\mathbb{E}$. Using the presentation of $\mathbb{E}$ in Example 2.2, we have that the coordinate axes have dimensions $\dim_{cc} \exp \operatorname{span} U_1 = \dim_{cc} \exp \operatorname{span} U_2 = 1$, $\dim_{cc} \exp \operatorname{span} V = 2$, and $\dim_{cc} \exp \operatorname{span} W = 3$. Among coordinate 2-dimensional spaces we have

$$\dim_{cc} \exp \operatorname{span}\{U_2, V\} = 3, \tag{8.2}$$

$$\dim_{cc} \exp \operatorname{span}\{X, Y\} = 4 \quad \text{if } \{X, Y\} = \{U_1, U_2\}, \{U_1, V\}, \{U_1, W\} \text{ or } \{U_2, W\}, \tag{8.3}$$

and

$$\dim_{cc} \exp \operatorname{span}\{V, W\} = 5, \tag{8.4}$$

while all coordinate hyperplanes have CC Hausdorff dimension 6. A more complete discussion of Hausdorff dimensions of and measures on submanifolds in $\mathbb{E}$ can be found in [46, §4] and [41]. The values in (8.2)–(8.4) may be verified as a straightforward application of [46, (1.4), (1.5) and (4.2)].

The preceding discussion together with Remark 6.5 show that Conjecture 8.1 is true in $\mathbb{E}$; compare the discussion in [41].

8.3. Hausdorff measure sharpness in the dimension comparison theorem

Establish the sharpness Theorem 2.6 *for on the level of Hausdorff measures.* More precisely, for each α and β with $\beta_-(\alpha) \leqslant \beta \leqslant \beta_+(\alpha)$, find a set $S \subset \mathbb{G}$ with $0 < \mathcal{H}_E^\alpha(S) < \infty$ and $0 < \mathcal{H}_{cc}^\beta(S) < \infty$. Our approach in Section 4 only provides such examples for a countable family of dimension pairs $(\alpha, \beta_-(\alpha))$. The almost sure dimension formulae in Theorem 2.8 hold for all dimension pairs but Theorem 2.8 does not provide any information about the Hausdorff measures.

8.4. Topological structure of Carnot fractals

There are several natural topological questions which arise in connection with fractals in Carnot groups. For example, every iterated function system in $\mathbb{R}^2$ satisfying the open set condition lifts to iterated function systems in $\mathbb{H}^1$ which also satisfy the open set condition. This fact greatly simplifies the computation of dimensions of such fractals as it permits the use of Theorem 4.11. The preceding observation relies on the fact that the group law in $\mathbb{H}^1$ (or any two-step Carnot group) involves Euclidean affine maps. We do not know when a Euclidean IFS satisfying the open set condition in the first stratum of a Carnot group $\mathbb{G}$ lifts to an IFS in $\mathbb{G}$ which again satisfies the open set condition. Similarly, in [4] we showed that if the invariant set of an IFS in $\mathbb{R}^2$ is connected, then some lift to $\mathbb{H}^1$ is again connected, provided the contraction ratios of the defining maps are sufficiently small, and conversely, that lifts of IFS in $\mathbb{R}^2$ satisfying the technical *post-critical finiteness* condition are generically totally disconnected. Analogs of such results in more general groups remain to be established.

8.5. Exceptional sets

Estimate the size of the set of translation parameter vectors $\mathbf{P}$ *for which* $\dim_{cc} K(\mathbf{P})$ *exceeds* $\beta_-(\dim_E K(\mathbf{P}))$ *by a definite amount.* It should be possible to use potential-theoretic arguments as in this paper to estimate the Hausdorff dimension (in either of the product metrics $(d_E)^M$ or $(d_{cc})^M$ on $\mathbb{G}^M$) of the set of vectors $\mathbf{P}$ for which $\dim_{cc} K(\mathbf{P}) \geqslant \beta_-(\dim_E K(\mathbf{P})) + \epsilon$, for fixed

$\epsilon > 0$. A similar result in the context of the almost sure dimension theory for Euclidean self-affine sets has recently been established by Falconer and Miao [19]. It may even be the case that the exceptional set

$$E = \{\mathbf{P} \in \mathbb{G}^M\colon \dim_{cc} K(\mathbf{P}) > \beta_-(\dim_E K(\mathbf{P}))\} \tag{8.5}$$

lies in a hypersurface. This is true, for instance, when $\mathbb{G} = \mathbb{H}^1$ and $M = 2$, as we now demonstrate.

Example 8.4. Let $\mathbf{r} = (r_1, r_2) \in (0, 1)^2$ with $r_1 + r_2 < 1$. Consider the invariant set $K(\mathbf{P})$ for $\{F_1, F_2\}$, $F_i(p) = p_i * \delta_{r_i}(p)$ as $\mathbf{P} = (p_1, p_2)$ varies in $\mathbb{H}^1 \times \mathbb{H}^1$. When $\pi_1(p_1) = \pi_1(p_2)$, $K(\mathbf{P})$ is a Cantor set lying along a translate of the x_3-axis, and satisfies $\dim_{cc} K(\mathbf{P}) = 2 \dim_E K(\mathbf{P})$. Otherwise, $K(\mathbf{P})$ is a horizontal set (in fact, a subset of a horizontal curve), and satisfies $\dim_{cc} K(\mathbf{P}) = \dim_E K(\mathbf{P})$. Thus in this case $E = \{(p_1, p_2)\colon \pi_1(p_1) = \pi_1(p_2)\}$, a hyperplane in $\mathbb{H}^1 \times \mathbb{H}^1$.

8.6. Carnot–Carathéodory manifolds

Extend the results of this paper to regular Carnot–Carathéodory manifolds. One approach to this question would be to reduce to the Carnot group situation by studying the regularity of the exponential map which provides local parameterizations of charts on the manifold M by Mitchell's approximating Carnot group [49]. If one could show that such map is locally bi-Lipschitz at regular points, the dimension comparison problem for M could be related to the corresponding problem for the approximating group. Unfortunately, such parameterizations are in general only known to be bi-Hölder continuous with exponent given by the reciprocal of the step, which is too weak to provide any nontrivial information about dimension comparison on M. Compare the discussion in Section 7.6 of [9].

These difficulties can be overcome in some situations. We indicate the solution to Problems 1.3 and 1.1 in the Martinet space $\mathbb{M}$ [50, §2.3, Chapter 3].

Example 8.5. We recall that $\mathbb{M}$ is the Carnot–Carathéodory manifold whose underlying space is $\mathbb{R}^3$ (we use coordinates $\tilde{p} = (\tilde{x}, \tilde{y}, \tilde{z})$) with horizontal distribution $H\mathbb{M}$ given as the span of the vector fields $\tilde{X} = \frac{\partial}{\partial \tilde{x}}$ and $\tilde{Y} = \frac{\partial}{\partial \tilde{y}} + \tilde{x}^2 \frac{\partial}{\partial \tilde{z}}$, or equivalently as the kernel of the defining form $\tilde{\omega} = d\tilde{z} - \tilde{x}^2\, d\tilde{y}$. We note the existence of a singular locus $\Sigma = \{\tilde{p}\colon \tilde{x} = 0\}$ in $\mathbb{M}$; the number of brackets required to span the full tangent space is equal to 2 at all points in $\mathbb{M} \setminus \Sigma$, but is equal to 3 at all points in Σ.

The CC metric on $\mathbb{M}$ is defined as for Carnot groups: $d_{cc}(\tilde{p}, \tilde{q})$ is the infimum of the lengths of all horizontal paths joining $\tilde{p}$ and $\tilde{q}$, where an absolutely continuous path $\gamma : [a, b] \to \mathbb{M}$ is horizontal if $\gamma'(t)$ lies in $H_{\gamma(t)}\mathbb{M}$ for almost every t, and the length is computed with respect to the fiberwise inner product on $H\mathbb{M}$ for which $\tilde{X}$ and $\tilde{Y}$ are an orthonormal basis.

Comparing Hausdorff dimensions of subsets S with respect to the sub-Riemannian and Euclidean dimensions on $\mathbb{M} = \mathbb{R}^3$, we find

$$\beta_-^{\mathbb{M}}(\dim_E S) \leqslant \dim_{cc} S \leqslant \beta_+^{\mathbb{M}}(\dim_E S), \tag{8.6}$$

where $\beta_-^{\mathbb{M}}(\alpha) = \max\{\alpha, 2\alpha - 2\}$ and $\beta_+^{\mathbb{M}}(\alpha) = \min\{3\alpha, \alpha + 2, 4\}$, see Fig. 8.

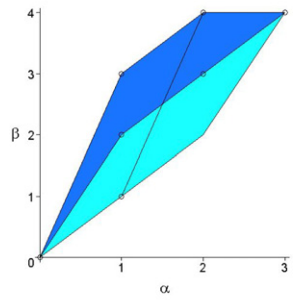


Fig. 8. Solutions to Problems 1.3 and 1.1 in $\mathbb{M}$.

To verify (8.6), we write $\mathbb{M} = \Omega_+ \cup \Sigma \cup \Omega_-$, where $\Omega_+ = \{\tilde{p}\colon \tilde{x} > 0\}$ and $\Omega_- = \{\tilde{p}\colon \tilde{x} < 0\}$. Equipped with the CC metric, each of the regions $\Omega_\pm$ is locally bi-Lipschitz equivalent with a domain in $\mathbb{H}^1$ (alternatively, $J^1(\mathbb{R},\mathbb{R})$), in fact, the map $p = (x,y,z) \mapsto (\sqrt{x}, y, z)$ is locally bi-Lipschitz from the domain $\{p \in J^1(\mathbb{R},\mathbb{R})\colon x > 0\}$ to $\Omega_+ \subset \mathbb{M}$. A simple computation shows that the CC metric in the singular locus Σ satisfies an estimate of the form

$$d_{cc}\big((0,\tilde{y}_1,\tilde{z}_1),(0,\tilde{y}_2,\tilde{z}_2)\big) \simeq |y_1 - y_2| + |z_1 - z_2|^{1/3}.$$

In effect, the solutions to Problems 1.3 and 1.1 in $\mathbb{M}$ can be obtained by combining the solutions in $\mathbb{H}^1$ and Σ. Using product sets and Fubini-type theorems for Hausdorff measure in $\mathbb{R}^2$, we obtain

$$\beta_-^S(\dim_E S) \leqslant \dim_{cc} S \leqslant \beta_+^S(\dim_E S), \quad \forall S \subset \Sigma,$$

where $\beta_-^S(\alpha) = \max\{\alpha, 3\alpha - 2\}$ and $\beta_+^S(\alpha) = \min\{3\alpha, \alpha + 2\}$. Since any set $S \subset \mathbb{M}$ can be decomposed in the form

$$S = (S \cap \Omega_+) \cup (S \cap \Sigma) \cup (S \cap \Omega_-),$$

we easily obtain (8.6). Unions of suitable examples in $\Omega_\pm$ and Σ show that the bounds $\beta_\pm^{\mathbb{M}}$ are sharp. Summarizing,

$$\Delta'(\mathbb{M}) = \mathrm{co}\big(\Delta'\big(\mathbb{H}^1\big) \cup \Delta'(\Sigma)\big),$$

where $\mathrm{co}(S)$ denotes the convex hull of S. Fig. 8 also shows the corresponding solution to Problem 1.1; we leave to the reader the identification of the relevant examples.

The preceding argument demonstrates the subtleties which arise for this problem in the singular locus, where the higher commutator relations are counterbalanced by the fact that such loci are typically of a smaller (Euclidean) dimension.

8.7. *Other metric spaces*

The goal of this paper has been the study of measure and dimension comparison for two compatible metrics on a common space with the aim of quantifying the degree to which sub-Riemannian metrics are non-Riemannian. It would be interesting to identify other situations where similar considerations arise. Analysis on postcritically finite self-similar fractals presents itself as a natural candidate. We refer to [39] and [60] for introductions to this fascinating subject. Other examples to consider could include the boundaries of various Gromov hyperbolic spaces equipped with their visual metrics. The Gromov boundaries of certain hyperbolic buildings I_{pq}, introduced by Bourdon [10,11], and later studied by Bourdon and Pajot [12,13], provide another source of metric measure spaces with good first-order analytic properties. Note that each of these spaces is homeomorphic with the Menger curve.

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